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# General Gravity in the Transversal Physics

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Abstract. From the abstract mathematical point of view, modern new transversal physics is based on transversal sets theory. In this sense, we shall show that the translation and rotation plays an important role in modern new physics.

A first culminating point was the discovery of the laws of planetary motions by the Prague astronomer and mathematician Johannes Kepler (1571–1639) during the years from 1609 to 1619.

Newton based his work on Kepler's results and Galilei's (1564–1642) observation that all bodies fall at the same rate, i.e., receive a *constant acceleration.*[1]

Already in 1802 Newton's theory of gravity was a great triumph. One year earlier Piazzi, in Palermo, discovered the planetoid Ceres as a star of magnitude eight and was able to follow its orbit for 9 degrees before losing it. The young Gauss (1777–1855) then computed the entire orbit by employing new methods of the calculus of observations; and using this result, Olbers rediscovered Ceres in 1802.

Today we know that the motion of the perihelion cannot be explained with Newton's theory of gravity, but is a consequence of the general theory of relativity, which was developed by Einstein in 1915. From this theory the above value follows very accurately. In this sense I give an affirmative answer that velocities bigger than the velocity of light $c$ by Nikola Tesla in 1932 – exist.

In the preceding sense I based the general transversal gravity theory on a new transversal min-max theory which I give in the last part of the paper.

**First fact of Transversal Physics**: *There exist in some spaces of physics some velocities which are bigger of the velocity of the light c.* Main facts of transversal physics are gravitational uneven functions and equations of the general transversal gravity.

2010 *Mathematics Subject Classification.* Primary: 04A25, 51M16, 65F35, 54E15, 05A16, 47H10; Secondary: 54E35, 54H25.

*Key words and phrases.* Gravity, general gravity, Kepler's results, Newton's theory of gravity, Einsten's equations, results by Nikola Tesla, forms of the second Kepler's law (on the sides of the space), gravity in the general convex (concave) algebra, gravity in the middle algebra, $n$-body problem, basic uneven equations of the transversal physics, Nonlinear relativistic physics.

## 1. Fundamental Elements of the Transversal Physics

The groundwork for the architecture of this book and new physics is founded on the following objects: transversal sets, transversal linear spaces (upper, lower and middle), general convexity, general concavity, general convex linearity, general concave linearity, gravitational uneven functions, basic uneven equations of Transversal Physics, transversal and min-max points. The following illustration is indicative:

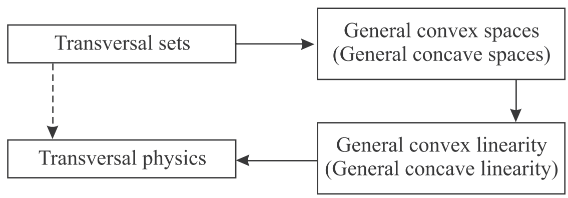


Figure 1

In this paper, for the first time, you can view an entirety new physics under the name: "Transversal Physics". This new physics is a new mathematical theory which is founded from objects of: transversal sets, transversal spaces, general convexity, general concavity, transversal points, min-max points, rotation and translation. There objects are groundwork of it's architecture.

The transversal physics is based from the object of transversal set and its technology. In this sense, every set has three part (or three sides, or three projections) as parts which are not see but this sides *de facto* existing as three (upper, lower and medial) transversal sets.

Related to the above, every space has its three sides which are not in the classical union. The sides of the space are connected in some other sense.

The new transversal physics extension comprehends all three main physics until now it known: Newton's physics, Einstein's physics (=Special Theory of Relativity) and Einsten's General Theory of Relativity.

A great achievement of the Transversal Physics is possibility that the velocity $v$ in an arbitrary general convex (concave) space is bigger of the velocity of light $c$, i.e., $|v| < |G(c, c, \lambda)|$ or $|v| < |R(c, c, \lambda)|$, where $G$ and $R$ are given general convex and general concave structures respectively.

This book provides an essential introduction to the ideas, methods, and applications of transversal physics theory. This theory includes: transversal minimax theory, transversal measures and integration, general convex functions, transversal physics, new gravitation theory, general convexity, general concavity, transversal functional analysis, and applications of transversal spaces. The following illustration is essential:

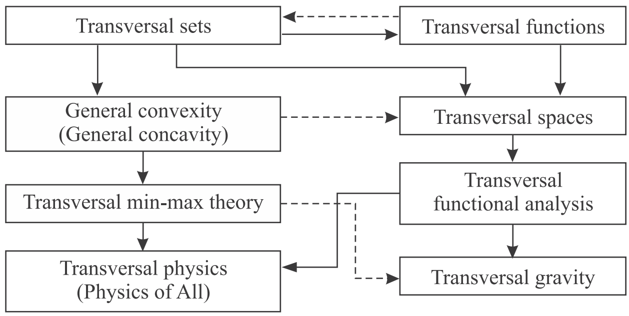


Figure 2

In the middle transversal physics (=Einstein's physics) the velocity $|v| < c$, meanwhile in the other two physics on the general convex and the general concave sides the velocity $v$ can be bigger of the velocity of light $c$.

Adequate to the technology of a transversal space (=general convex, general concave and middle sides of the space) three physics exist which are foundation under the corresponding algebras as: general convex physics, general concave physics, and middle physics! These physics are what else formulated similar, meanwhile they are enough different in the consequences which production as and geometrically facts which from there hold.

The recent experiments in the Laboratory for Physics, known as CERN (=European Organization for Nuclear Research) at Geneva-Switzerland, confirm one of three physics of Newton and Einstein are not to declare, however they also confirm the correctness of the foundation of the all merciful Transversal Physics. **First fact of Transversal Physics**: *There exist in some spaces of physics some velocities which are bigger of the velocity of the light* $c$. Their velocities are possibly on every sides of space. The velocity $v > c$ it can't be in Einstein's physics!

Since the tunnel at CERN pictures a transversal linear space (adequate, has three own sides: upper, lower and middle) in the preceding sense, if the particle "neutrino" go only at the middle side of the space-tunnel, then the velocity of the particle is $v < c$ adequate to the middle algebra of the physics space.

However, since the tunnel is a transversal chaos space, thus the particle *neutrino* can be go under different sides of the physics space in the different algebras. Adequate to the preceding facts we learn that the velocities at the linear (general convex or general concave) space of the transversal physics can be bigger of the velocity of light $c$, i.e., $|v| < |G(c, c, \lambda)|$ or $|v| < |R(c, c, \lambda)|$!

**Open problem.** *If the calculations are not correct in the Laboratory for Physics CERN, to work out a new experiment for to make a note of a new velocity* $v$ *in the Transversal Physics in the form* $G(c, c, \lambda)$ *or* $R(c, c, \lambda)$ *which* *is bigger of the velocity of light* $c$!? (In connection with this fact see: Interview of Nikola Tesla (by John J. A. O'Neill, Science Editor of the Eagle)[1] in 1932 (10 July-76th Birthday) from Brooklyn Eagle: *Tesla cosmic ray motor may transmit power'round earth.*)

**General convex relativistic physics**. Further, in this section for physics, let $X$ be a form as $X \cup \mathbb{R}$ (or $X \cup \mathbb{C}$) and $Y = \mathbb{R}$ (or $Y = \mathbb{C}$). Consider two systems $\sigma$ and $\sigma'$ with corresponding space elements of general convex structures $G(x, x, \lambda)$, $G(y, y, \lambda)$, $G(z, z, \lambda)$, $G(x', x', \lambda)$, $G(y', y', \lambda)$ and $G(z', z', \lambda)$. Assume also that $\sigma$ and $\sigma'$ are two **transversal convex inertial systems** of the form as on sketch with corresponding system times $G(t, t, \lambda)$ and $G(t', t', \lambda)$.

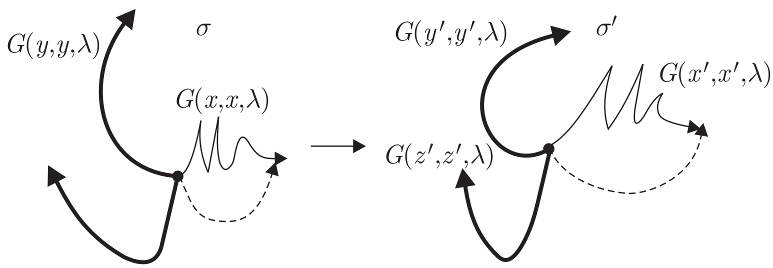


FIGURE 3

In this sense, a system $\sigma$ is a *transversal general convex system* precisely if there exists a system time $G(t, t, \lambda)$ for it such that each mass point, which is far away enough from other masses and shielded against fields, e. g., light pressure, remains at rest or moves rectilinearly with constant velocity.

At the beginning of this section I formulated the following three postulates in the corresponding form as:

(A) All transversal general convex inertial systems are physically equivalent, i.e., physical processes are the same in all transversal general convex inertial systems when initial boundary conditions are the same.

[1]**...Exceed Velocity of Light.** "All of my investigations seem to point to the conclusion that they are small particles, each carrying so small a charge that we are justified in calling them neutrons. They move with great velocity, exceeding that of light."

"More than 25 years ago I began my efforts to harness the cosmic rays and I can now state that I have succeeded in operating a motive device by means of them."

"I was able to prevail upon Dr. Tesla to give me some idea of the principle upon which his cosmic ray motor works."

"I will tell you in the most general way", he said. "The cosmic ray ionizes the air, setting free many chargesions and electrons. These charges are captured in a condenser which is made to discharge through the circuit of the motor."

**Hopes to Build Large Motor.** "I have hopes of building my motor on a large scale, but circumstances have not been favorable to carrying out my plan."

(B) (*Constant velocity of light*). In every transversal general convex inertial system, light travels with the same constant velocity $G(c, c, \lambda)$ in every direction, where $c$ is the speed of light.

(C) (*Principle of translation*). There exists a transversal general convex inertial system. If $\sigma$ is a transversal general convex inertial system, then also each transversal general convex system $\sigma'$, which is obtained from $\sigma$ by constant translatory motion, is a transversal general convex inertial system.

In base the second chapter of the book (*Fundamental elements of the Transversal Physics*), the change from $\sigma$ to $\sigma'$ on the upper side of the space is given by the *special Tasković transformations* in the form as

$$G(x, x, \lambda) = \frac{G(x', x', \lambda) + vG(t', t', \lambda)}{\sqrt{1 - v^2/G(c, c, \lambda)^2}}, \tag{1}$$

and, to obtain an equation allowing to find the value of $G(t, t, \lambda)$ according to the known values of $G(x', x', \lambda)$ and $G(t', t', \lambda)$ let us delete the element $G(x, x, \lambda)$ and corresponding solve the resulting expression relative to $G(t, t, \lambda)$, we obtain

$$G(t, t, \lambda) = \frac{G(t', t', \lambda) + vG(x', x', \lambda)}{\sqrt{1 - v^2/G(c, c, \lambda)^2}}, \tag{2}$$

where $c$ is the velocity of light and $|v| < |G(c, c, \lambda)|$ for a general convex (affine) structure $G(x, x, \lambda) \in \mathbb{R}$ (or $\mathbb{C}$) in general convex (affine) algebra.

The combination of equations $G(y, y, \lambda) = G(y', y', \lambda)$, $G(z, z, \lambda) = G(z', z', \lambda)$, (1) and (2) is called **general convex (affine) transformations** of general convex (affine) spaces.

If we solve these equations of general convex (affine) transformations relative to the primed quantities, we get the equations for transformations from the frame $\sigma$ to $\sigma'$ in the following form

$$\begin{aligned} G(x', x', \lambda) &= \frac{G(x, x, \lambda) - vG(t, t, \lambda)}{\sqrt{1 - v^2/G(c, c, \lambda)^2}} \\ G(y', y', \lambda) &= G(y, y, \lambda), G(z', z', \lambda) = G(z, z, \lambda), \\ G(t', t', \lambda) &= \frac{G(t, t, \lambda) - vG(x, x, \lambda)}{\sqrt{1 - v^2/G(c, c, \lambda)^2}}. \end{aligned} \tag{3}$$

As it should be expected with a view to the equal rights of the frames $\sigma$ and $\sigma'$, equations (3) differ from their counterparts of general convex transformations only in the sign of $v$.

It is easy to understand that when $|v| < |G(c, c, \lambda)|$, the general convex transformations become the same as the *Galilean type* ones for general convex spaces. The latter thus retain their importance for speeds that are small in comparison with the speed of light in a vacuum.

When $|v| > |G(c, c, \lambda)|$, the equations of general convex linear transformations and (3) for $G(x, x, \lambda)$, $G(t, t, \lambda)$, $G(x', x', \lambda)$ and $G(t', t', \lambda)$ become imaginary. This agrees with the fact that motion at a speed exceeding that of light in a vacuum is impossible. For $v = G(c, c, \lambda)$ we can systems $\sigma$ and $\sigma'$ return in the origin positions.

In connection with the previous, for $|v| < |G(c, c, \lambda)|$ the change from $\sigma'$ and $\sigma$ is given by the *special general convex transformations* in the following form as

$$\begin{aligned} G(x, x, \lambda) &= G(x', x', \lambda) + vG(t', t', \lambda), \\ G(y, y, \lambda) &= G(y', y', \lambda), \\ G(z, z, \lambda) &= G(z', z', \lambda), \\ G(t, t, \lambda) &= G(t', t', \lambda) + vG(x', x', \lambda), \end{aligned} \tag{4}$$

where equations (4) allow us to pass over from elements and time measured in the frame $\sigma'$ to those measured in the frame $\sigma$.

If we solve equations (4) relative to the primed quantities elements, we get the equations for transformation from the frame $\sigma$ to $\sigma'$ in the following form as

$$\begin{aligned} G(x', x', \lambda) &= G(x, x, \lambda) - vG(t, t, \lambda), \\ G(y', y', \lambda) &= G(y, y, \lambda), \\ G(z', z', \lambda) &= G(z, z, \lambda) \\ G(t', t', \lambda) &= G(t, t, \lambda) - vG(x, x, \lambda). \end{aligned}$$

The **general concave (lower affine) relativistic physics** is formulated on the lower side of the space in the proper manner via general concave (lower affine) algebra with general concave (lower affine) structure $R(x, x, \lambda) \in \mathbb{R}$ (or $\mathbb{C}$). In this sense, we have correspondent general concave (lower affine) transformations of the general concave (lower affine) space for the velocity $|v| < |R(c, c, \lambda)|$. Also, for small velocities (as a physics of the Galilean type transformations), we obtain a form of the special general concave (lower affine) transformations.

**Middle linear spaces**. In the preceding two part of this paper I have had two spaces (or two sides of a space): general convex linear space and general concave linear space. As a new space (or as *third side of a given space*) is a middle linear space in 2005 by T a s k o v i ć. In this sense, a **middle linear space** is a general convex linear space and a general concave linear space simultaneous.

As an important example of a middle linear space we have a classical linear (vector) space.

**Annotation.** From the facts in 2005 by T a s k o v i ć we have a main result of the form: *that every space, de facto, has three sides*; which in this case I denoted with as: general convex linear space, general concave linear space and middle linear space.

Further considerations of the middle linear spaces we esteem all the preceding facts on general convex and general concave linear spaces.

**Main Consequence**. From the facts of the preceding two parts of this book we directly obtain complete Einstein's relativistic theory as a relativistic physics on the middle transversal linear spaces.

In this sense, indeed, for small velocities on the middle linear space for $G(x, x, \lambda) = R(x, x, \lambda) = x$, we obtain the *Galilean transformations.* In this case, also, we have so-called *special Lorentz Transformations.*

In the global physics of all (*Transversal Physics, General convexity and General concavity*), we first introduce fundamental elements of a new "general convex" minimax theory which unifies and connects three known theories on fixed point, transversality and von Neumann's minimax theory.

## 2. Gravity on the sides of the space

The ingenuity of astronomers and astrophysicists, who gave gatherd our present knowledge about the iniverse, is admirable, and should shame all master detectives in the world's literature.

A first culminating point was the discovery of the laws of planetary motions by the Prague astronomer and mathematician Johannes Kepler (1571–1639) during the years from 1609 to 1619.

At this time Kepler studied a tremendous a mount of numerical data, which had been collected by Tycho Brahe (1564–1601). The laws are:

1) (First law). *The planets move on elliptic orbits with the sun at one focus.*
2) (Second law). *The line segment joining a planet and the sun sweeps out equal areas in equal times.*
3) (Third law). *The squares of the periods of revolution of two planets about the sun are proportional to the cubes of the semimajor axes of the ellipses.*

These laws are true with respect to a Cartesian coordinate system with the sun at its origin and whose axes are firmly connected with the stellar sky. A vague impression of Kepler's tremendous scientific achievement is obtained by noting that all the numerical data originated from the moving planet Earth.

The step was never taken by the great astronomers of antiquity. For example, it was important to observe that Kopernikus' (1473–1543) hypothesis about a circular orbit of the planet Mars let to an average error of 8 minutes of arc. Laws 1) to 3) are of a kinematic nature, i.e., they describe the motion, but not its cause. Isaac Newton's (1643–1727) path-breaking idea was then to recognize that 1) to 3) follows from a universal law (*law of gravity*) and a *general equation of motion.* Newton based his work on Kepler's results and

Galilei's (1564–1642) observation that all bodies fall at the same rate, i.e., receive a *constant acceleration*.[2]

Already in 1802 Newton's theory of gravity was a great triumph. One year earlier Piazzi, in Palermo, discovered the planetoid Ceres as a star of magnitude eight and was able to follow its orbit for 9 degrees before losing it. The young Gauss (1777–1855) then computed the entire orbit by employing new methods of the calculus of observations; and using this result, Olbers rediscovered Ceres in 1802.

Today we known that the motion of the perihelion cannot be explained with Newton's theory of gravity, but is a consequence of the general theory of relativity, which was developed by Einstein in 1915.

From this theory the above value follows very accurately.

**Gravity in the general convex (affine) algebra.** Consider the coordinate system as at Figure 4 with the corresponding general convex (affine) space.

Therefore, $G(r,r,\lambda)$ and $G(r_0,r_0,\lambda)$ can be related only by expressions (from the former general convex (affine) linear transformations) of the kind

$$G(r,r,\lambda) = \alpha G(r_0,r_0,\lambda), \tag{5}$$

where $\alpha$ is a constant; thus we obtain the following form of the preceding equality as

$$\frac{\mathrm{d}G(r,r,\lambda)}{\mathrm{d}t} = \frac{d\alpha}{\mathrm{d}t} G(r_0,r_0,\lambda) + \alpha \frac{\mathrm{d}G(r_0,r_0,\lambda)}{\mathrm{d}t}, \tag{6}$$

[2]**History of facts.** This last observation led Newton to assume that on the surface of the earth a gravitational force exists which causes the free fall. Because of the interactions between the planets, 1) to 3) are only approximately true. Until 1781 the only known planets were Mercury, Venus, Earth, Mars, Jupiter, and Saturn.

In 1781, Herschel discovered Uranus with a telescope and in 1811, in connection with a prize problem of the Paris Academy, Delambre collected numerical data about the motion of Uranus. It was noticed that for several time during the previous one hundred years, Uranus had been registered as a fixed star. But the observed and the calculated data did not completely match each other. It was expected that the deviations were caused by a still unknown planet.

In 1845 and 1846, after long and complicated computations two young astronomers, E. Adams (1849–1892) and F. Leverrier (1811–1877), indepentently found the orbit of a new planet, called Neptune. Later, in 1846 Galle (1812–1910) at the Berlin astronomical observatory discovered it by the following the numerical data contined in a letter by Leverrier.

Jacobi (1804–1851) then wrote: "One can only admire, how it is possible to obtain such precise results from so few and uncertain results. Those who call this discovery accidental should also be encouraged to make such accidental discoveries themselves."

In 1930, the planet Pluto was discovered at the Flagstaff astronomical observatory in Arizona (U.S.A.) as a result of a perturbation calculation for Neptune.

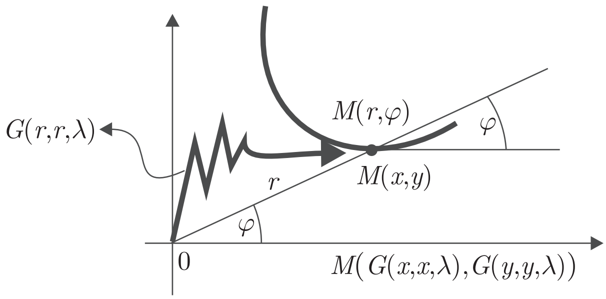


FIGURE 4

where the left side of this equality is the *velocity on the upper side of space* (or general convex (affine) structure of the velocity denoted by $v_{us}$), i.e.,

$$v_{us} := \frac{\mathrm{d}G(r, r, \lambda)}{\mathrm{d}t}$$

of the object $M(G(x, x, \lambda), G(y, y, \lambda))$ which is in the motion as one Figure 4. In the same manner we obtain that the expression $\mathrm{d}G(r_0, r_0, \lambda)/\mathrm{d}t$ is the velocity on the upper side of space for the object (convex (affine) structure) $G(r_0, r_0, \lambda)$ of the point $(r_0, \varphi)$. Then, from (6), we have

$$\frac{\mathrm{d}G(r, r, \lambda)}{\mathrm{d}t} = v_{u\alpha} G(r_0, r_0, \lambda) + v_{un} G(n_0, n0), \lambda), \tag{7}$$

where $v_{u\alpha} = \mathrm{d}\alpha/\mathrm{d}t$ and $v_{un} = \alpha \mathrm{d}G(\varphi, \varphi, \lambda)/\mathrm{d}t$, such that $v_{u\alpha}$ and $v_{un}$ are corresponding unit general convex (affine) structures, i.e., $v_{u\alpha}$ and $v_{un}$ are components of the velocity $v_{us}$. Difference of (6) via $\mathrm{d}t$ give

$$\begin{aligned}\frac{\mathrm{d}^2 G(r, r, \lambda)}{\mathrm{d}t^2} = {} & \frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} G(r_0, r_0, \lambda) + \frac{\mathrm{d}\alpha}{\mathrm{d}t}\frac{\mathrm{d}G(r_0, r_0, \lambda)}{\mathrm{d}t} + \\ & + \frac{\mathrm{d}\alpha}{\mathrm{d}t}\frac{\mathrm{d}G(\varphi, \varphi, \lambda)}{\mathrm{d}t} G(n_0, n_0, \lambda) + \alpha \frac{\mathrm{d}^2 G(\varphi, \varphi, \varphi)}{\mathrm{d}t^2} G(n_0, n_0, \lambda) + \\ & + \alpha \frac{\mathrm{d}G(\varphi, \varphi, \lambda)}{\mathrm{d}t} \cdot \frac{\mathrm{d}G(n_0, n_0, \lambda)}{\mathrm{d}t},\end{aligned}$$

where $\mathrm{d}G(n_0, n_0, \lambda) = (-\mathrm{d}G(\varphi, \varphi, \lambda)/\mathrm{d}t) G(r_0, r_0, \lambda)$, i.e., hence we obtain the following equality in the form as

$$\begin{aligned}\frac{\mathrm{d}^2 G(r, r, \lambda)}{\mathrm{d}t^2} = {} & \left[\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} - \alpha \left(\frac{\mathrm{d}G(\varphi, \varphi, \lambda)}{\mathrm{d}t}\right)^2\right] G(r_0, r_0, \lambda) + \\ & + \left[\alpha \frac{\mathrm{d}^2 G(\varphi, \varphi, \lambda)}{\mathrm{d}t^2} + 2 \frac{\mathrm{d}\alpha}{\mathrm{d}t}\frac{\mathrm{d}G(\varphi, \varphi, \lambda)}{\mathrm{d}t}\right] G(n_0, n_0, \lambda),\end{aligned} \tag{8}$$

where the left side of this equality is the **acceleration** on the upper side of space (or general convex (affine) structure of the acceleration denoted by

$a_{us}$), i.e.,

$$a_{us} := \frac{\mathrm{d}^2 G(r,r,\lambda)}{\mathrm{d}t^2} = a_{\alpha u} G(r_0,r_0,\lambda) + a_{nu} G(n_0,n_0,\lambda), \tag{9}$$

where are components of acceleration $a_{us}$ on the upper side of the space given in the following form as

$$a_{\alpha u} := \frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} - \alpha\left(\frac{\mathrm{d}G(\varphi,\varphi,\lambda)}{\mathrm{d}t}\right)^2, \tag{10}$$

$$a_{nu} := \alpha\frac{\mathrm{d}^2 G(\varphi,\varphi,\lambda)}{\mathrm{d}t^2} + 2\frac{\mathrm{d}\alpha}{\mathrm{d}t}\frac{\mathrm{d}G(\varphi,\varphi,\lambda)}{\mathrm{d}t} = \frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\left(\alpha^2\frac{\mathrm{d}G(\varphi,\varphi,\lambda)}{\mathrm{d}t}\right) \tag{11}$$

Applying the preceding facts to the planets move, if the origin with Figure 4 is to equal the sun, from equation for the second Kepler's law, for $\alpha \equiv G(\alpha,\alpha,\lambda)$ we obtain

$$G(\alpha,\alpha,\lambda)^2\frac{\mathrm{d}G(\varphi,\varphi,\lambda)}{\mathrm{d}t} = \text{Constant} := C, \tag{12}$$

i.e., we have the **form of the second Kepler law** on the upper side of space (or *general convex (affine) structure of the second Kepler's law*). Thus, from (11) and (12) we obtain $a_{nu} = 0$, which means that planet, in every point of proper trajectory has only acceleration on general convex (affine) structure in way which is connection with the sun. Further from (10) we obtain

$$a_{\alpha u} = -\frac{C^2}{G(\alpha,\alpha,\lambda)^2}\left[\frac{1}{G(\alpha,\alpha,\lambda)} + \frac{\mathrm{d}^2}{\mathrm{d}G(\varphi,\varphi,\lambda)^2}\left(\frac{1}{G(\alpha,\alpha,\lambda)}\right)\right], \tag{13}$$

i.e., we have this **form of the Biné equality** on the upper side of space; thus, from (13) and from the first Kepler's law we obtain

$$a_{\alpha u} = -\frac{aC^2}{b^2}\cdot\frac{1}{G(\alpha,\alpha,\lambda)^2} = -4\pi^2 k\frac{1}{G(\alpha,\alpha,\lambda)^2}$$

where $k := a^3/T^2$, $C := 2\pi ab/T$, $a$ and $b$ are half-axis' of an ellipse, the parameter of the ellipse $p = b^2/a$, and $T$ is the time planet passing round the sun. Then, for $\mu = 4\pi^2 k = 4\pi^2 a^3/T^2$ we obtain $a_{\alpha u} = -\mu/G(\alpha,\alpha,\lambda)^2$, i.e., from (9) and $a_{nu} = 0$ we have

$$a_{us} = -\frac{\mu}{G(\alpha,\alpha,\lambda)^2}G(r_0,r_0,\lambda), \tag{14}$$

i.e., this equation means that every planet, in every own position, has the preceding *form of acceleration on the upper side of space* to direct toward the sun. Further, to the multiplication of (14) with $G(m_1,m_1,\lambda)$ for the mass of the planet $P_1$ on upper side of space, we obtain an *upper gravitational force* $\mathcal{R} = a_{us}G(m_1,m_1,\lambda)$ acts from $P_1$ onto the sun $P_2$ in the form

$$\mathcal{R} = -\eta\frac{G(m_1,m_1,\lambda)}{G(\alpha,\alpha,\lambda)^2}G(r_0,r_0,\lambda). \tag{15}$$

According to Newton's principle of actio=ractio, we find that, conversely, the upper gravitational force $-\mathcal{R}$ acts from $P_2$ onto $P_1$, i.e., from (15) final, we obtain the **upper gravitational force** in the form as

$$\mathcal{R}_{us} = \mu \frac{G(m_1, m_1, \lambda)G(m_2, m_2, \lambda)}{G(\alpha, \alpha, \lambda)^2}, \tag{16}$$

where $\mu$ is an upper universal constant and is called *upper gravitational constant* on the upper side of the space. Thus, the upper gravitational force has the *upper potential* in the form as

$$P_{us} := -\mu \frac{G(m_1, m_2, \lambda)G(m_2, m_2, \lambda)}{G(\alpha, \alpha, \lambda)}.$$

**Gravity in the general concave (lower affine) algebra.** From the preceding facts, $R(r, r, \lambda)$ and $R(r_0, r_0, \lambda)$ can be related only by expressions (from the former general concave (lower affine) linear transformations) of the kind

$$R(r, r, \lambda) = \alpha R(r_0, r_0, \lambda), \tag{17}$$

where $\alpha$ is a constant; thus we obtain the following form of the preceding equality as

$$\frac{\mathrm{d}R(r, r, \lambda)}{\mathrm{d}t} = \frac{\mathrm{d}\alpha}{\mathrm{d}t} R(r_0, r_0, \lambda) + \alpha \frac{\mathrm{d}R(r_0, r_0, \lambda)}{dt}, \tag{18}$$

where the left side of this equality is the *velocity on the lower side of space* (or general concave (lower affine) structure of the velocity denoted by $v_{ls}$), i.e.,

$$v_{ls} := \frac{\mathrm{d}R(r, r, \lambda)}{\mathrm{d}t}$$

of the object $M(R(x, x, \lambda), R(y, y, \lambda))$ which is in the motion as one Figure 5. In the same manner we obtain that the expression $dR(r_0, r_0, \lambda)/dt$ is the velocity on the upper side of space for the object (convex (affine) structure) $R(r_0, r_0, \lambda)$ of the point $(r_0, \varphi)$. Then, from (18), we have

$$\frac{dR(r, r, \lambda)}{dt} = v_{l\alpha} R(r_0, r_0, \lambda) + v_{ln} R(n_0, n_0), \lambda), \tag{19}$$

where $v_{l\alpha} = \mathrm{d}\alpha/\mathrm{d}t$ and $v_{ln} = \alpha \mathrm{d}R(\varphi, \varphi, \lambda)/\mathrm{d}t$, such that $v_{l\alpha}$ and $v_{ln}$ are corresponding unit general convex (affine) structures, i.e., $v_{l\alpha}$ and $v_{ln}$ are components of the velocity $v_{ls}$. Difference of (18) via d$t$ give

$$\begin{aligned}\frac{\mathrm{d}^2 R(r, r, \lambda)}{\mathrm{d}t^2} = {} & \frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} R(r_0, r_0, \lambda) + \frac{\mathrm{d}\alpha}{\mathrm{d}t} \frac{\mathrm{d}R(r_0, r_0, \lambda)}{dt} + \\ & + \frac{\mathrm{d}\alpha}{\mathrm{d}t} \mathrm{d} \frac{\mathrm{d}R(\varphi, \varphi, \lambda)}{\mathrm{d}t} R(n_0, n_0, \lambda) + \alpha \frac{\mathrm{d}^2 R(\varphi, \varphi, \varphi)}{\mathrm{d}t^2} R(n_0, n_0, \lambda) + \\ & + \alpha \frac{\mathrm{d}R(\varphi, \varphi, \lambda)}{\mathrm{d}t} \cdot \frac{\mathrm{d}R(n_0, n_0, \lambda)}{\mathrm{d}t},\end{aligned}$$

where $\mathrm{d}R(n_0, n_0, \lambda) = (-\mathrm{d}R(\varphi, \varphi, \lambda)/\mathrm{d}t)R(r_0, r_0, \lambda)$, i.e., hence we obtain the following equality in the form as

$$\begin{aligned}\frac{\mathrm{d}^2R(r,r,\lambda)}{\mathrm{d}t^2} &= \left[\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} - \alpha\left(\frac{\mathrm{d}R(\varphi,\varphi,\lambda)}{\mathrm{d}t}\right)^2\right]R(r_0,r_0,\lambda)+\\ &\quad+\left[\alpha\frac{\mathrm{d}^2R(\varphi,\varphi,\lambda)}{\mathrm{d}t^2} + 2\frac{\mathrm{d}\alpha}{\mathrm{d}t}\frac{\mathrm{d}R(\varphi,\varphi,\lambda)}{\mathrm{d}t}\right]R(n_0,n_0,\lambda),\end{aligned} \tag{20}$$

where the left side of this equality is the **acceleration** on the lower side of space (or general concave (lower affine) structure of the acceleration denoted by $a_{ls}$), i.e.,

$$a_{ls} := \frac{\mathrm{d}^2R(r,r,\lambda)}{\mathrm{d}t^2} = a_{\alpha l}R(r_0,r_0,\lambda) + a_{lu}R(n_0,n_0,\lambda), \tag{21}$$

where are components of acceleration $a_{ls}$ on the upper side of the space given in the following form as

$$a_{\alpha l} := \frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} - \alpha\left(\frac{\mathrm{d}R(\varphi,\varphi,\lambda)}{\mathrm{d}t}\right)^2, \tag{22}$$

$$a_{lu} := \alpha\frac{\mathrm{d}^2R(\varphi,\varphi,\lambda)}{\mathrm{d}t^2} + 2\frac{\mathrm{d}\alpha}{\mathrm{d}t}\frac{\mathrm{d}R(\varphi,\varphi,\lambda)}{\mathrm{d}t} = \frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\left(\alpha^2\frac{\mathrm{d}R(\varphi,\varphi,\lambda)}{\mathrm{d}t}\right) \tag{23}$$

Applying the preceding facts to the planets move, if the origin with Figure 5 is to equal the sun, from equation for the second Kepler's law, for $\alpha \equiv R(\alpha, \alpha, \lambda)$ we obtain

$$R(\alpha,\alpha,\lambda)^2\frac{\mathrm{d}R(\varphi,\varphi,\lambda)}{\mathrm{d}t} = \text{Constant} := C, \tag{24}$$

i.e., we have the **form of the second Kepler law** on the lower side of space (or *general concave (lower affine) structure of the second Kepler's law*). Thus, from (23) and (24) we obtain $a_{nl} = 0$, which means that planet, in every point of proper trajectory has only acceleration on general concave (lower affine) structure in way which is connection with the sun. Further from (22) we obtain

$$a_{\alpha l} = -\frac{C^2}{R(\alpha,\alpha,\lambda)^2}\left[\frac{1}{R(\alpha,\alpha,\lambda)} + \frac{\mathrm{d}^2}{\mathrm{d}R(\varphi,\varphi,\lambda)^2}\left(\frac{1}{R(\alpha,\alpha,\lambda)}\right)\right], \tag{25}$$

i.e., we have this *form of the Biné equality* on the lower side of space; thus, from (25) and from the first Kepler's law we obtain

$$a_{\alpha l} = -\frac{aC^2}{b^2}\cdot\frac{1}{R(\alpha,\alpha,\lambda)^2} = -4\pi^2k\frac{1}{R(\alpha,\alpha,\lambda)^2}$$

where $k := a^3/T^2$, $C := 2\pi ab/T$, $a$ and $b$ are half-axis' of an ellipse, the parameter of the ellipse $p = b^2/a$, and $T$ is the time planet passing round the sun. Then, for $\mu = 4\pi^2 k = 4\pi^2 a^3/T^2$ we obtain $a_{\alpha l} = -\mu/R(\alpha,\alpha,\lambda)^2$, i.e., from (21) and $a_{nl} = 0$ we have

$$a_{ls} = -\frac{\mu}{R(\alpha,\alpha,\lambda)^2} R(r_0, r_0, \lambda), \tag{26}$$

i.e., this equation means that every planet, in every own position, has the preceding *form of acceleration on the lower side of space* to direct toward the sun. Further, to the multiplication of (26) with $R(m_1, m_1, \lambda)$ for the mass of the planet $P_1$ on lower side of space, we obtain an lower *gravitational force* $\mathcal{R} = a_{ls} G(m_1, m_1, \lambda)$ acts from $P_1$ onto the sun $P_2$ in the form

$$\mathcal{R} = -\eta \frac{R(m_1, m_1, \ldots)}{R(\alpha,\alpha,\lambda)^2} R(r_0, r_0, \lambda). \tag{27}$$

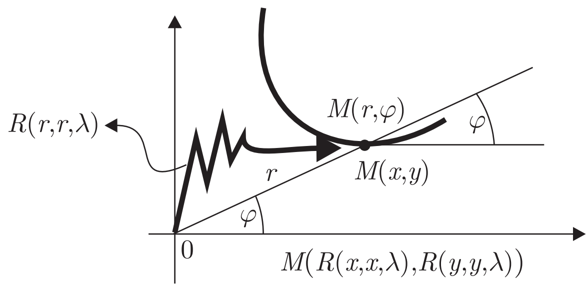


FIGURE 5

According to Newton's principle of actio=ractio, we find that, conversely, the lower gravitational force $-\mathcal{R}$ acts from $P_2$ onto $P_1$, i.e., from (27) final, we obtain the **lower gravitational force** in the form as

$$\mathcal{R}_{ls} = \mu \frac{R(m_1, m_1, \lambda) R(m_2, m_2, \lambda)}{R(\alpha,\alpha,\lambda)^2}, \tag{28}$$

where $\mu$ is an lower universal constant and is called *lower gravitational constant* on the lower side of the space. Thus, the lower gravitational force has the *lower potential* in the form as

$$P_{ls} := -\mu \frac{R(m_1, m_2, \lambda) R(m_2, m_2, \lambda)}{R(\alpha,\alpha,\lambda)}.$$

**Gravity in the middle algebra.** As a gravity on the middle side of the space of a given space is a middle gravity by Tasković [2009]. In this sense, we recall that the **middle gravitational force** is an upper gravitational force and a lower gravitational force simultaneous.

As an important example of a middle gravitational force we have the classical *Newton's Law of Gravitation* in the case $G(x,x,\lambda) = R(x,x,\lambda) = x$!

We notice that the direction of the preceding upper force shows that it is an attracting upper force (as Figure 6).

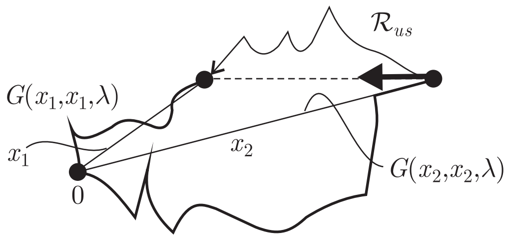


Figure 6

**The $n$-Body Problem.** We begin with the motion of $n$ mass points which are subject only to the gravitational force on the sides of the space (for example, sun $x_1$ and $n-1$ planets $x_2, \ldots, x_n$).

According to laws of gravity on the sides of the space, for the masses $m_1, \ldots, m_n$ of the corresponding celestial bodies we obtain the **equations of $n$-body problem** on the upper side of the space in the form as

$$G(m_i, m_i, \lambda)\frac{\mathrm{d}^2 G(r_i, r_i, \lambda)}{\mathrm{d}t^2} = \sum_{k=1}^{n} \mu \frac{G(m_i, m_i, \lambda)G(m_k, m_k, \lambda)}{d_{ik}^3}\eta_{ik}$$

for $i = 1, \ldots, n$; where $d_{ik} := |G(r_k, r_k, \lambda) - G(r_i, r_i, \lambda)|$ and $\eta_{ik} := G(r_k, r_k, \lambda) - G(r_i, r_i, \lambda)$ for $i = 1, \ldots, n$. On the other hand, as in the preceding case, on the lower side of the space we obtain the **equations of $n$-body problem** in the form as

$$R(m_i, m_i, \lambda)\frac{\mathrm{d}^2 R(r_i, r_i, \lambda)}{\mathrm{d}t^2} = \sum_{k=1}^{n} \mu \frac{R(m_i, m_i, \lambda)R(m_k, m_k, \lambda)}{d_{ik}^3}\eta_{ik}$$

for $i = 1, \ldots, n$; where $d_{ik} := |R(r_k, r_k, \lambda) - R(r_i, r_i, \lambda)|$ and $\eta_{ik} := R(r_k, r_k, \lambda) - R(r_i, r_i, \lambda)$ for $i = 1, \ldots, n$.

**The $n$-Body Problem in the middle algebra.** According to middle algebra, for the masses $m_1, \ldots, m_n$ of the corresponding celestial bodies we have the *classical equations of $n$-body problem* in the case $G(x, x, \lambda) = R(x, x, \lambda) = x$.

We notice that the direction of the preceding lower force shows that it is an attracting lower force (as on Fig. 7).

**Gravitational uneven equilibrium on the sides of the space.** The stability problem of the planetary system in general form is as follows: Is it possible that the planets collide, fall into the sun, drastically change their orbits, or as an extremal situation leave the solar system? Also, are small perturbations, e.g., caused by cosmic dust, able to cause one of the preceding cases?

This is a question of existential consequences, because already small perturbations of the orbit of the earth would have catastrophic consequences

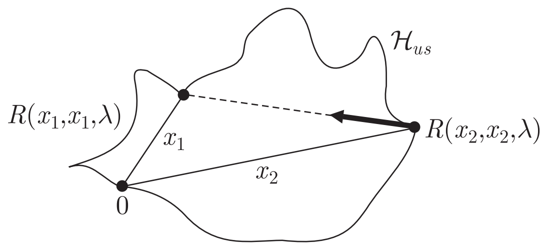


Figure 7

for the life of our planet. The great interest in explicit analytic solutions for the $n$-body problem was motivated by the hope that through this the stability problem could be solved. The complete answer is still open!

In this sense, from the preceding facts and via the results of the second chapter of this book, on the upper side of the space, we have that the **gravitational uneven upper functions** are in the form as

$$u(g_{\mu\lambda}) := \max_{x,y\in X} \inf \big\{G(x,x,\mu), G(y,y,\lambda), g\big(G(x,x,\mu), G(y,y,\lambda)\big)\big\}$$

and

$$u(h_{\mu\lambda}) := \min_{x,y\in X} \sup \big\{G(x,x,\mu), G(y,y,\lambda), g\big(G(x,x,\mu), G(y,y,\lambda)\big)\big\},$$

with the general convex (affine) algebra $Y = \mathbb{R}$, where $g : \mathbb{R}^2 \to \mathbb{R}$. If $u(g_{\mu\lambda}) = u(h_{\mu\lambda})$, then on the upper side of the space (in general convex (affine) algebra) there exists so-called **balance** or **gravitational upper equilibrium**.

On the other hand, on the lower side of the space, we have that the **gravitational uneven lower functions** are in the form as

$$l(g_{\mu\lambda}) := \max_{x,y\in X} \inf \Big\{R(x,x,\mu), R(y,y,\lambda), d\big(R(x,x,\mu), R(y,y,\lambda)\big)\Big\}$$

and

$$l(h_{\mu\lambda}) := \min_{x,y\in X} \sup \Big\{R(x,x,\mu), R(y,y,\lambda), d\big(R(x,x,\mu), R(y,y,\lambda)\big)\Big\},$$

with the general concave (lower affine) algebra $Y = \mathbb{R}$, where $d : \mathbb{R}^2 \to \mathbb{R}$. If $l(g_{\mu\lambda}) = l(h_{\mu\lambda})$, then on the lower side of the space (in the general concave (lower affine) algebra) there exists so-called **uneven balance** or **gravitational lower uneven equilibrium**.

Also, if $G(x,x,\lambda) = R(x,x,\lambda) = \lambda x$ (:=*rotation*) or $G(x,x,\lambda) = R(x,x,\lambda) = x + \lambda$ (:=*translation*), then we have **gravitational middle uneven functions** $m(g_{\mu\lambda})$ and $m(h_{\mu\lambda})$ on the middle side of the space. If $m(g_{\mu\lambda}) = m(h_{\mu\lambda})$, then on the middle side of the space there exists so-called **uneven balance** or **gravitational middle uneven equilibrium**.

Otherwise, it is of great importance the states of gravity and planets behavior on the edges of the space (=where the space sides collide!). In this sense, we have the following **gravitational edges uneven functions** (on the edges of general convexity (affinity) and general concavity (lower affinity)) in the form as

$$e(g_{\mu\lambda}) := \max_{x,y\in X} \inf \Big\{G(x,x,\mu), R(y,y,\lambda), \psi\big(G(x,x,\mu), R(y,y,\lambda)\big)\Big\}$$

and

$$e(h_{\mu\lambda}) := \min_{x,y\in X} \sup \Big\{G(x,x,\mu), R(y,y,\lambda), \psi\big(G(x,x,\mu), R(y,y,\lambda)\big)\Big\},$$

where $\psi : \mathbb{R}^2 \to \mathbb{R}$. If $e(g_{\mu\lambda}) = e(h_{\mu\lambda})$, then on the edges of two sides of the space from general convexity (lower affinity) and general concavity (affinity) there exists so-called **uneven balance** or **gravitational edge uneven equilibrium**.

**General annotation.** In general convex (affine) and general concave (lower affine) algebras for $Y = \mathbb{C}$, also hold the preceding formulas (as on the edge algebra) for gravitational uneven functions when only $G(x,x,\lambda)$ and $R(x,x,\lambda)$ will to change with $|G(x,x,\lambda)|$ and $|R(x,x,\lambda)|$ respectively.

**Basic Uneven Equations of the Transversal Physics.** The *basic uneven equations* of the new Transversal Physics which determine via the sides of the space and via: general convex (affine), general concave (lower affine), middle and edge uneven balance are

$$\text{(Be)}\qquad \begin{aligned} u(g_{\mu\lambda}) &= u(h_{\mu\lambda}), & l(g_{\mu\lambda}) &= l(h_{\mu\lambda}),\\ m(g_{\mu\lambda}) &= m(h_{\mu\lambda}), & e(g_{\mu\lambda}) &= e(h_{\mu\lambda}); \end{aligned}$$

with the solutions through $G(x,x,\lambda)$, $R(x,x,\lambda)$, and $g, d, \psi : \mathbb{R}^2 \to \mathbb{R}$. For example, we can to write down the equation $m(g_{\mu\lambda}) = m(h_{\mu\lambda})$ in the following form

$$\text{(jc2)}\qquad \max_{x,y\in\mathbb{R}\setminus\{0\}} \inf \big\{x, y, g(x,y)\big\} = \min_{x,y\in\mathbb{R}\setminus\{0\}} \sup \big\{x, y, g(x,y)\big\}$$

where $g(x,y) = 2^{-1}(K_m^2/x + K_m^2/y)$ for $K_m := K_{ij}^m \in \mathbb{R} \setminus \{0\}$ and $m = 1,2,3,4$ as an equation in the form

(29)

$$\max_{x,y\in\mathbb{R}\setminus\{0\}} \inf \left\{x, y, \frac{1}{2}\left(\frac{K_m^2}{x} + \frac{K_m^2}{y}\right)\right\} = \min_{x,y\in\mathbb{R}\setminus\{0\}} \sup \left\{x, y, \frac{1}{2}\left(\frac{K_m^2}{x} + \frac{K_m^2}{y}\right)\right\},$$

for $m = 1,2,3,4$; then this equation, from Theorem 1, has at least one solution $\xi$ if and only if $\xi = K_m$ for the arbitrary constants $K_m := K_{ij}^m \in \mathbb{R} \setminus \{0\}$ and $m = 1,2,3,4$. Thus for the metric tensors $(K_1, K_2, K_3, K_4) := K_{\mu\lambda} = 2R^{-1}(R_{\mu\lambda} - \tau T_{\mu\lambda})$, with the universal constant $\tau = 8\pi G/c^4$ and with the gravitational constant $G$, we obtain on middle side of the space the **General Theory of Relativity** by Einstein [1916], i.e., we obtain *Einstein's equations* in the form as

$$m(g_{\mu\lambda}) = 2R^{-1}(R_{\mu\lambda} - \tau T_{\mu\lambda}).$$

Taking one consideration with another, from the preceding facts of $m(g_{\mu\lambda}) = m(h_{\mu\lambda})$ and (29), *de facto*, we have the continuum middle physics on the middle side of the space.

On the other hand, applying Theorem 1 to the uneven equation $u(g_{\mu\lambda}) = u(h_{\mu\lambda})$ we obtain that its holds if and only if there exist the points $x_0, y_0, r_0, z_0 \in X$ such that

$$\inf\Big\{G(x_0,x_0,\mu), G(y_0,y_0,\lambda), g\big(G(x_0,x_0,\mu), G(y_0,y_0,\lambda)\big)\Big\} =$$
$$= \sup\Big\{G(r_0,r_0,\mu), G(z_0,z_0,\lambda), g\big(G(r_0,r_0,\mu), G(z_0,z_0,\lambda)\big)\Big\}.$$

## 3. Foundation of the general gravity[3]

In this section we introduced fundamental elements of a new convex minimax theory which unified and connected three known theories on fixed points, transversality and von Neumann's minimax theory.

In classical von Neumann's minimax theory fundamental notions is saddle point. In new convex minimax theory its role plays transversal point.

This important fact was discovered by John von Neumann in 1937, who established a coincidence statement in $\mathbb{R}^n$ and made a direct use of it in the proof of his well-known Minimax Principle. In this sense, in 1928 von Neumann investigated the concept of a saddle point for a mapping $f : A \times B \to \mathbb{R}$, where $A$ and $B$ are nonempty sets. A point $(x_0, y_0) \in A \times B$ is called a *saddle point* of $f : A \times B \to \mathbb{R}$ if

$$\max_{x\in A}\min_{y\in B} f(x,y) = \min_{y\in B}\max_{x\in A} f(x,y),$$

i.e., in an equivalent form, if the following inequalities hold

$$f(x_0,y) \leq f(x_0,y_0) \leqslant f(x,y_0) \quad \text{for all} \quad (x,y) \in A \times B.$$

[3]**Historical facts.** In the letter of April 27 in 1984 sent to me by American-Japanese mathematician Ky Fan about my work *Minimax theorems on ordered sets* (that was not published at that time) literally there are following lines: *I have read the paper and found it very interesting. The paper is based on some quite original ideas. I would suggest that you submit the paper to a different journal (for example, Mathematische Zeitschrift, Math. Annalen, Trans. Amer. Math. Soc., etc.). Again I want to say that I like very much the original ideas in your paper.*

The fact is that there is a completely new (it did not exist until now) minimax-theory on lattices, with completely new geometric shapes as objects; lattices theory opens a completely new area, significant for further research. This fact was very important for the work of Garrett Birkhoff, which even he himself mentioned during lecture named *New convex minimax theory on lattices*, which he held on Technische Hochschule Darmstadt in June 1991 on the occasion of his 80th birthday. By the way, I spoke of this publicly for the first time along my published work in 1984 in Varna, although the beginnings of this problem can be found in my doctoral thesis in 1978.

Since then, geometrical problems of a similar kind (as well as their analytic counterparts) have attracted broad attention and remarkable progress has been made both in generalization of the original results as well in finding new applications in a variety of mathematical areas, see: Aubin [1977] and Ky Fan [1972].

In connection with this, the antipodal statement of Borsuk in 1938 and the theorem on topological tranversality occupy central position in nonlinear analysis. Many interesting statements have been proved about continuous real-valued functions defined on an $n$-sphere $\mathbb{S}^n$.

In 1945 A. W. Tucker (see also S. Lefshetz [1930]) gave discovered a very interesting combinatorial lemma. By its use, he has given elementary and elegant proofs of various well-known topological properties of the $n$-sphere, such as the antipodal-point theorem of Borsuk-Ulam, that of Lusternik-Schnirelmann, and many others.

In this section we introduced a new convex minimax theory which unified and connected three preceding theories of fixed points, transversality and von Neumann's minimax theory.

In connection with the preceding, in 1987 we considered a concept of transversal points, for the mapping $f$ of a nonempty set $X$ into a partial ordered set $P$, in the sense, that $f$ has a *transversal point* $\xi \in P$ if there is a decreasing function $g : P^2 \to P$ such that the following equality holds

(T)

$$\max_{x,y\in X} \min \Big\{ f(x), f(y), g(f(x), f(y)) \Big\} = \min_{x,y\in X} \max \Big\{ f(x), f(y), g(f(x), f(y)) \Big\} := \xi.$$

In this section, we also consider some other points of this type. Applications in nonlinear functional analysis, specially, in minimax theory and convex analysis are considered.

**Fundamental elements of a new minimax theory.** Let $P := (P, \preccurlyeq)$ be a partially ordered set by the ordering relation $\preccurlyeq$. The function $g : P^k \to P$ ($k$ is a fixed positive integer) is **decreasing** on the ordered set $P$ if $a_i, b_i \in P$ and $a_i \preccurlyeq b_i$ $(i = 1, \dots, k)$ implies $g(b_1, \dots, b_k) \preccurlyeq g(a_1, \dots, a_k)$.[4]

[4]**History of games theory.** The history of games theory is quite tangled. In this note I give destitution some open questions.

The first important result of this theory is Zermelo's lecture presented in 1912 to the Fifth International Congress of Mathematicians (see: E. Zermelo, Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels, Proc. of the Fifth International Congress of Mathematicians, Cambridge, **2**(1912), 501–504).

Zermelo's result is the determinateness of such games as chess. World War I is the reason why Zermelo's paper failed to be appreciated according to its merit.

The second step in 1921 (next in 1924) is due to Émile Borel. He found the minimax concept after World War I, but his papers were not duly appreciated in France and failed to reach Polish mathematicians.

In connection with this, in 1925 Hugo Steinhaus was to define the best strategies in a game older than chess and perhaps older than human civilization: the game of chase and escape (see: H. Steinhaus, *Definicje potrzebne do teorji gry i pościgu*, Myśl Academicka, Lwów 1925, nr 1. Reprinted in English, Naval Res. Logist. Quart., **7**(1960), 105-107). He gave a minimax principle for the special game of "chase and escape" in its

Let $L$ be a lattice and $g$ a mapping from $L^2$ into $L$. For any $g : L^2 \to L$ it is natural to consider the following property of **local comparability**, which means, if $\xi \in L$ is comparable with $g(\xi,\xi) \in L$ then $\xi$ is comparable with every $t \in L$.

**Lemma 1** (Sup-Inf Inequalities)**.** *Let $L := (L, \preccurlyeq)$ be a lattice and let $g : L^2 \to L$ be a decreasing mapping. If $L$ has a property of local comparability, then for arbitrary functions $p : X \to L$ and $q : Y \to L$ ($X$ and $Y$ are arbitrary nonempty sets) the following relations are valid:*

$$\text{(S)} \qquad \xi \preccurlyeq g(\xi,\xi) \quad \textit{implies} \quad \xi \preccurlyeq \sup\Big\{p(x), q(y), g(p(x), q(y))\Big\},$$

*and*

$$\text{(I)} \qquad g(\xi,\xi) \preccurlyeq \xi \quad \textit{implies} \quad \inf\Big\{p(x), q(y), g(p(x), q(y))\Big\} \preccurlyeq \xi,$$

*for all $x \in X$ and for all $y \in Y$. Hence, in particular, $\xi = g(\xi,\xi)$ implies that the following inequalities hold in the form as*

$$\text{(U)} \qquad \inf\Big\{p(x), q(y), g(p(x), q(y))\Big\} \preccurlyeq \xi \preccurlyeq \sup\Big\{p(x), q(y), g(p(x), q(y))\Big\},$$

*for all $x \in X$ and for all $y \in Y$.*

---

simplest form. It can be written as follows:

$$\text{(Z)} \qquad \min_{\alpha}\max_{\beta} T\Big(\alpha(A,B), \beta(A,B)\Big) \le \max_{\beta}\min_{\alpha} T\Big(\alpha(A,B), \beta(A,B)\Big),$$

for the real time $T$, where $\alpha(A,B)$ being the angle between East and the course of $A$, and an analogous definition gives $\beta(A,B)$ as the angle between East and the course of $B$. The functions $\alpha$ and $\beta$ are called the *strategies* of partners $A$ and $B$, respectively.

We notice that the proof of (Z) is easy: It is evident that the real time $T$ resulting from arbitrary choice of strategies $\alpha$ and $\beta$ lies between the left and right side of inequality (Z). From Steinhaus in 1925 “general” means here a theory of pursoit in a limited plane or on arbitrary surfaces such as an ellipsoid or a torus.

J. Von Neumann was avare of the importance of the minimax principle in 1928. It is, however, difficult to understand the absence of a quotation of Zermelo’s lecture in his publications.

Jan Mycielski in 1964 has found a few years ago a formulation of Zermelo’s theorem as a formula which is one of several connected with the name of Augustus de Morgan in 1847 and belongs to formal logic.

One of von Neumann’s important achievements in the theory of games is his theorem about closing open games. Ryll-Nardzewski’s version in 1965 of this result assumes that there are teams of players led by captains who are responsible for the decisions of their teams.

S. Banach and S. Mazur in 1925 resolved to investigate infinite games to see if the minimax rule in a form applies to such alternating games with perfect information; also, in 1925 they found such games, refuting by their discovery conjecture that all alternating games with perfect information are closed.

H. Steinhaus in 1960 have chosen the idea of replacing Zermelo’s Axiom of Choice by the following **Axiom of Determinacy**: *All two-person alternating games with perfect information are closed.*

Steinhaus’s collaboration with Jan Mycielski started at this point. It had such episodes as two telegrams (Berkeley – Zakopane) in summer 1961: “The axiom is dead”, and a day later: “Axiom still living”. Later in 1964 J. Mycielski has been an application of Axiom of Determinacy for the Banach-Tarski decomposition of the ball.

An immediate consequence (special case for totally ordered sets) of the preceding Lemma 1 is its following form.

**Lemma 2** (Min-Max Inequalities)**.** *Let $P$ be a total ordered set by the order relation $\preccurlyeq$, and let $g : P^2 \to P$ be a decreasing mapping. Then for functions $p, q : X \to P$ ($X$ is a nonempty set) the following relations are valid:*

$$\xi \preccurlyeq g(\xi,\xi) \quad \textit{implies} \quad \xi \preccurlyeq \max\Big\{p(x), q(y), g(p(x), q(y))\Big\},$$

*and*

$$g(\xi,\xi) \preccurlyeq \xi \quad \textit{implies} \quad \min\Big\{p(r), q(s), g(p(r), q(s))\Big\} \preccurlyeq \xi,$$

*for all $x, y, r, s \in X$. Hence, in particular, $\xi = g(\xi,\xi)$ implies that the following inequalities hold in the form as*

$$\min\Big\{p(r), q(s), g(p(r), q(s))\Big\} \preccurlyeq \xi \preccurlyeq \max\Big\{p(x), q(y), g(p(x), q(y))\Big\},$$

*for all $x, y, r, s \in X$.*

Quantifying the assertions (S), (I) and (U) we notice that we obtain the following interesting conclusions (which, incidentally are their equivalent formulations for $X = Y$):

(ES) $$\xi \preccurlyeq g(\xi,\xi) \quad \text{implies} \quad \xi \preccurlyeq \inf_{x,y\in X} \sup\Big\{p(x), q(y), g(p(x), q(y))\Big\},$$

and

(EI) $$g(\xi,\xi) \preccurlyeq \xi \quad \text{implies} \quad \sup_{x,y\in X} \inf\Big\{p(x), q(y), g(p(x), q(y))\Big\} \preccurlyeq \xi,$$

and $g(\xi,\xi) = \xi$ implies the following inequalities:
(EU)
$$\sup_{x,y\in X} \inf\Big\{p(x), q(y), g(p(x), q(y))\Big\} \preccurlyeq \xi \preccurlyeq \inf_{x,y\in X} \sup\Big\{p(x), q(y), g(p(x), q(y))\Big\}.$$

On the other hand, we note, that it is easy to construct a decreasing mapping on a complete lattice which is not a total ordered set, but the property of local comparability is fulfilled, see Figure 8.

**Example 1.** Let $L$ be the lattice on Figure 8 and let $g : L \to L$ be defined by $g(0) = 1$, $g(a) = b$, $g(b) = a$, $g(c) = 0$, $g(1) = 0$. Evidently, $g$ is a decreasing and the property of local comparability is fulfilled, but the set $L$ is not totally ordered.

**Remark 1.** The above statements (Lemma 4) still hold when $g : L^k \to L$ ($k$ is a fixed positive integer) is a decreasing function. The proof is quite similar; the assertion corresponding to (S) and (I) look as follows

(S') $$\xi \preccurlyeq g(\xi,\dots,\xi) \quad \text{implies} \quad \xi \preccurlyeq \sup\Big\{\lambda_1,\dots,\lambda_k, g(\lambda_1,\dots,\lambda_k)\Big\},$$

and

(I') $$g(\xi,\dots,\xi) \preccurlyeq \xi \quad \text{implies} \quad \inf\Big\{\lambda_1,\dots,\lambda_k, g(\lambda_1,\dots,\lambda_k)\Big\} \preccurlyeq \xi,$$

for arbitrary functions $\lambda_1,\dots,\lambda_k : X \to L$, where $X$ is an arbitrary nonempty set. Also, in particular, $\xi = g(\xi,\dots,\xi)$ implies

(U') $$\inf\Big\{\lambda_1,\dots,\lambda_k, g(\lambda_1,\dots,\lambda_k)\Big\} \preccurlyeq \xi \preccurlyeq \sup\Big\{\lambda_1,\dots,\lambda_k, g(\lambda_1,\dots,\lambda_k)\Big\},$$

for arbitrary functions $\lambda_i \in X$ $(i = 1, \dots, k)$, where $X$ is an arbitrary nonempty set. To simplify the notation we will give the proof only for the case $k = 2$.

*Proof of Lemma 4. Implication* (S). Let $\xi \preccurlyeq g(\xi, \xi)$ and $\lambda = \sup\{p(x), q(y)\}$, where the elements $x \in X$ and $y \in Y$ are arbitrarily chosen. If $\xi \preccurlyeq \lambda$, then

$$\xi \preccurlyeq \sup\Big\{p(x), q(y), g(p(x), q(y))\Big\} \quad \text{for all} \quad x \in X \text{ and } y \in Y, \tag{30}$$

obviously holds. If $\lambda \preccurlyeq \xi$, then $x \preccurlyeq g(x, x) \preccurlyeq g(p(x), q(y))$ and (30) holds, too. We see that the comparability of elements $\xi$ and $\lambda$ is possible as a consequence of the property of local comparability.

One gets the implication (I) by applying the above result to the case where the relation $\preccurlyeq$ is replaced by the relation $\succcurlyeq$; in fact, after this change, every supremum becomes an infimum and the function $g$ remains decreasing with respect to each argument. Thus, we have (I). The last assertion (U) is evident. The proof is complete. □

**Lemma 3** (Tasković, [1978]). *Let $P$ be a total ordered set by the order relation $\preccurlyeq$, and let $g : P^2 \to P$ be a decreasing mapping. Then, the following conditions are equivalent:*

$$\min\{t, g(t, t)\} \preccurlyeq \xi \preccurlyeq \max\{t, g(t, t)\} \quad \textit{for all} \quad t \in P, \tag{M}$$

*and the following condition*

$$\xi = \min P(g \preccurlyeq) \quad \textit{or} \quad \xi = \max P(\preccurlyeq g), \tag{EM}$$

*where $P(g \preccurlyeq) := \{t \in P \mid g(t, t) \preccurlyeq t\}$ and $P(\preccurlyeq g) := \{t \in P \mid t \preccurlyeq g(t, t)\}$.*

We note, as a direct consequence of this assertion that the following facts hold in the form as:

1) The number of points $\xi \in P$ with characteristic (M) can be $0, 1$ or $2$.

Besides that:

2) Every one of these cases can be realized.
3) Especially, if $P$ in the meaning of order is an everywhere dense set of points, the number of points with characteristic (M) is 0 or 1, and
4) If the set $P$ has the characteristic of density (:=that is for every Dedekind's cross section the lower class has a maximum or the upper class has a minimum) the number of points is 1 or 2.
5) If $\xi \in P$ is the fixed point of the mapping $g : P^2 \to P$, then $\xi$ is the point with characteristic (M), and then (M) holds if and only if

$$\max_{x \in P} \min\{x, g(x, x)\} = \min_{x \in P} \max\{x, g(x, x)\} = \xi.$$

**Further remarks.** In Lemma 5 the assumption that $P := (P, \preccurlyeq)$ is totally ordered cannot be replaced by the weaker assumption that $P$ is a lattice. More precisely, (M) implies (EM) holds true in the case of any poset, while (EM) implies (M) is in general false even for lattices. Indeed, from (M) it follows that each element $t \in P$ is comparable with $g(t, t)$ so that $\xi \in P(g \preccurlyeq)$ or $\xi \in P(\preccurlyeq g)$. In the first case $t \in P(g \preccurlyeq)$, i.e., $g(t, t) \preccurlyeq t$; so we have $\xi \preccurlyeq \max\{t, g(t, t)\} = t$, and hence $\xi = \min P(g \preccurlyeq)$. A symmetric proof shows that $\xi \in P(\preccurlyeq g)$ implies $\xi = \max P(\preccurlyeq g)$.

On the other hand, the structure on Figure 9 is obviously a lattice and the function $g : P \to P$ defined by $g(a) = c$, $g(b) = g(d) = b$, $g(c) = a$, where $P =$

$\{a, b, c, d\}$, is decreasing. In this case we have also $P(g \preccurlyeq) = \{b, c\}$, $P(\preccurlyeq g) = \{a, b\}$, and thus $b = \min P(g \preccurlyeq) = \max P(\preccurlyeq g)$, i.e., (EM) holds. However, (M) is false since $d$ is not comparable with $b = g(d)$.

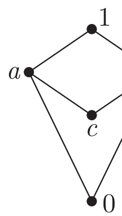


Figure 8

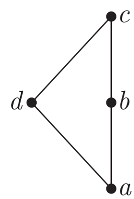


Figure 9

*Proof of Lemma 3.* (EM) *implies* (M). Let $\xi = \min P(g \preccurlyeq)$ or $\xi = \max P(\preccurlyeq g)$. Now, let $x \in P(\preccurlyeq g)$, $y \in P(g \preccurlyeq)$ and $y \prec x$. Then $g(y, y) \preccurlyeq y \prec x \preccurlyeq g(x, x)$, i.e., $g(y, y) \preccurlyeq g(x, x)$ is in contradiction with the decreasing of the function $g$. This means that for all $x \in P(\preccurlyeq g)$ and $y \in P(g \preccurlyeq)$ it follows that $x \preccurlyeq y$. Let $\xi = \max P(\preccurlyeq g)$; then if $t \in P(\preccurlyeq g)$ we have $t \preccurlyeq \xi$ and thus $\min\{t, g(t, t)\} \preccurlyeq x$ and then $\max\{t, g(t, t)\} = g(x, x) \succcurlyeq g(\xi, \xi) \succcurlyeq \xi$. If $t \in P(\preccurlyeq g)$, we have $\xi \preccurlyeq t$, and thus $\xi \preccurlyeq \max\{t, g(t, t)\}$. For $\xi \prec t$ we have $g(t, t) \preccurlyeq g(\xi, \xi) \preccurlyeq \xi$, i.e., $\min\{t, g(t, t)\} \preccurlyeq \xi$. The case $\xi = \min P(g \preccurlyeq)$ is symmetrical with the previous one.

(M) *implies* (EM). Let the point $\xi \in P$ have characteristics (M). Then $x \in P(\preccurlyeq g)$ implies $x \preccurlyeq g(x, x)$, that is, $x = \min\{x, g(x, x)\} \preccurlyeq \xi$, and $x \in P(g \preccurlyeq)$ implies $g(x, x) \preccurlyeq x$, that is, $x = \max\{x, g(x, x)\} \succcurlyeq \xi$. Then for all $x \in P(\preccurlyeq g)$ is $\xi \preccurlyeq x$, and for all $x \in P(g \preccurlyeq)$ is $\xi \preccurlyeq x$. Accordingly, as for all $x \in P(\preccurlyeq g)$ and $y \in P(g \preccurlyeq)$ relation $x \preccurlyeq y$ holds, we have the following: if $\xi \in P(\preccurlyeq g)$, then $\xi = \max P(\preccurlyeq g)$; if $\xi \in P(g \preccurlyeq)$, then $\xi = \min P(g \preccurlyeq)$. Owing to that, if the point $b \succ \xi$ satisfies the condition (M), then we must have $\xi = \max P(\preccurlyeq g)$, $b = \min P(g \preccurlyeq)$, and there cannot be any third point with characteristic (M). The proof is complete. $\square$

**Some comments.** The following example proves that two different points with characteristic (M) may exist: $P = \{a, b\}$, $a \prec b$, $g(a) = b$, $g(b) = a$. In that case both points $a$ and $b$ have characteristic (M). But, if $(P, \preccurlyeq)$ is an everywhere dense set (i.e., for $x \prec y$ there is $z \in P$ with $x \prec z \prec y$ for all $x, y \in P$), then there can be at most one point of characteristic (M).

Let us now give an example which shows that the points with characteristic (M) may not be fixed points. Let the mapping $g$ be defined by $g(x) = 1$ $(0 \le x \le 1/2)$ and $g(x) = 0$ $(1/2 < x \leqslant 1)$. Then on the set $P = [0, 1]$ the point $\xi = 1/2$ has characteristic (M), but it is not fixed point of the mapping $g : [0, 1] \to [0, 1]$.

With the help of the preceding statements, we now obtain the following fundamental fact of this section.

**Theorem 1** (Sup-Inf Theorem)**.** *Let $L := (L, \preccurlyeq)$ be a lattice and let $g : L^2 \to L$ be a decreasing mapping. If $L$ has a property of local comparability, then for some arbitrary functions $p : X \to L$ and $q : X \to L$ ($X$ is an arbitrary nonempty set)* *the equality*

$$\text{(SI)} \qquad \max_{x,y\in X} \inf\Big\{p(x), q(y), g(p(x), q(y))\Big\} = \min_{x,y\in X} \sup\Big\{p(x), q(y), g(p(x), q(y))\Big\}$$

*holds if and only if*

$$\text{(Si)} \qquad \inf\Big\{p(x_0), q(y_0), g(p(x_0), q(y_0))\Big\} = \sup\Big\{p(r_0), q(z_0), g(p(r_0), q(z_0))\Big\}$$

*for some* $x_0.y_0, r_0, z_0 \in X$.

*Proof.* The trivial fact that the strict inequality cannot hold in (EU) follows at once from (EU) and Lemma 1.

In this sense, the necessity of the condition being trivial, we only prove its sufficiency. If (Si) holds, then we have the following relations

$$\text{(31)} \qquad p(r_0), q(z_0), g(p(r_0), q(z_0)) \preccurlyeq s = i \preccurlyeq \inf\Big\{p(x_0), q(y_0), g(p(x_0), q(y_0))\Big\},$$

for $s := \sup\{p(r_0), q(z_0), g(p(r_0), q(z_0))\}$, $i := \inf\{\{p(x_0), q(y_0), g(p(x_0), q(y_0))\}$, and for some $x_0, y_0, r_0, z_0 \in X$. Since $g : L^2 \to L$ is decreasing, from (31) we obtain

(31')

$$g(i,i) = g(s,s) \preccurlyeq g(p(r_0), q(z_0)) \preccurlyeq s = i \preccurlyeq g(p(x_0), q(y_0)) \preccurlyeq g(s,s) = g(i,i),$$

i.e., $i = s = g(i,i) = g(s,s)$. Applying Lemma 1 (the case (U)) from local comparability we have

$$\inf\Big\{p(x), q(y), g(p(x), q(y))\Big\} \preccurlyeq i = s \preccurlyeq \sup\Big\{p(x), q(y), g(p(x), q(y))\Big\}$$

for all $x, y \in L$. Therefore, we have (SI). The proof is complete. □

An immediate consequence (special case) of the preceding statement is the following principle.

**Theorem 2** (Minimax Principle)**.** *Let $P$ be a total ordered set by the order relation $\preccurlyeq$, and let $g : L^2 \to L$ be a decreasing mapping. Then for some arbitrary functions $p : X \to P$ and $q : X \to P$ ($X$ is an arbitrary nonempty set) the equality*

(MM)

$$\max_{x,y\in X} \min\Big\{p(x), q(y), g(p(x), q(y))\Big\} = \min_{x,y\in X} \max\Big\{p(x), q(y), g(p(x), q(y))\Big\}$$

*holds if and only if*

$$\text{(Mm)} \qquad p(x_0) = q(y_0) := \xi = g(\xi,\xi) \quad \textit{for some} \quad x_0, y_0 \in X.$$

The statement above still holds when $g : P^k \to P$ ($k$ is a fixed positive integer). The proof is quite similar. Therefore, let $P$ be a total ordered set by the order relation $\preccurlyeq$, and $g : P^k \to P$ ($k \in \mathbb{N}$ is fixed) be a decreasing mapping. Then, the equality

(Uk)

$$\max_{\lambda_1,\dots,\lambda_k\in P} \min\Big\{\lambda_1, \dots, \lambda_k, g(\lambda_1, \dots, \lambda_k)\Big\} = \min_{\lambda_1,\dots,\lambda_k\in P} \max\Big\{\lambda_1, \dots, \lambda_k, g(\lambda_1, \dots, \lambda_k)\Big\}$$

holds if and only if

$$\lambda_1(a_1) = \ldots = \lambda_k(a_k) := \xi = g(\xi, \dots, \xi) \quad \text{for some} \quad a_1, \dots, a_k \in X,$$

where $\lambda_i : X \to P$ ($i = 1, \dots, k$) are arbitrary functions and $X$ is a nonempty set.

We remark that when $X = P$, $p(x) = x$ and $q(y) = y$ Theorem 43 reduces to that of our following former result.

**Corollary 1** (Tasković, [1978]). *Let $P$ be a total ordered set by the order relation $\preccurlyeq$, and let $g : P^2 \to P$ be a decreasing mapping. Then the equality*

$$\max_{x,y \in P} \min\{x, y, g(x,y)\} = \min_{x,y \in P} \max\{x, y, g(x,y)\}$$

*holds if and only if there is $\xi \in P$ such that $g(\xi, \xi) = \xi$.*

In connection with the preceding, we note that we can give an extension of the preceding Theorem 42, as a direct consequence of the preceding facts, in the following sense.

**Theorem 3** (General Sup-Inf Theorem). *Let $L := (L, \preccurlyeq)$ be a lattice and let $g : L^2 \to L$ be a mapping. Then for some arbitrary $p : X \to L$ and $q : X \to L$ ($X$ is an arbitrary nonempty set) the following equality holds*

$$\text{(SI')} \quad \max_{x,y \in X} \inf \Big\{p(x), q(y), g(p(x), q(y))\Big\} = \min_{x,y \in X} \sup \Big\{p(x), q(y), g(p(x), q(y))\Big\}$$

*if and only if the following inequalities hold*

$$\text{(DI)} \quad \begin{aligned} &\inf \Big\{p(x), q(y), g(p(x), q(y))\Big\} \preccurlyeq \inf \Big\{p(x_0), q(y_0), g(p(x_0), q(y_0))\Big\} = \\ &= \sup \Big\{p(r_0), q(z_0), g(p(r_0), q(z_0))\Big\} \preccurlyeq \sup \Big\{p(x), q(y), g(p(x), q(y))\Big\} \end{aligned}$$

*for some $x_0, y_0, r_0, z_0 \in X$ and for all $x, y \in X$.*

On the other hand, if $L$ is a total ordered set, then the condition (DI) is an equivalent with the following equality

$$\max_{x,y \in X} \min \Big\{p(x), q(y), g(p(x), q(y))\Big\} = \min_{x,y \in X} \max \Big\{p(x), q(y), g(p(x), q(y))\Big\}.$$

Also, in connection with the preceding equality (Uk), if $g : P^k \to P$ ($k$ is a fixed positive integer) is not decreasing mapping, we can extend equality (Uk). In this sense, if $g : P^k \to P$ ($k$ is a fixed positive integer) some arbitrary mapping then equality (Uk) holds if and only if the following inequalities hold

$$\begin{aligned} &\min\Big\{\lambda_1, \ldots, \lambda_k, g(\lambda_1, \ldots, \lambda_k)\Big\} \preccurlyeq \min\Big\{\lambda_1(a_1), \ldots, \lambda_k(a_k), g(\lambda_1(a_1), \ldots, \lambda_k(a_k))\Big\} = \\ &= \max\Big\{\lambda_1(b_1), \ldots, \lambda_k(b_k), g(\lambda_1(b_1), \ldots, \lambda_k(b_k))\Big\} \preccurlyeq \max\Big\{\lambda_1, \ldots, \lambda_k, g(\lambda_1, \ldots, \lambda_k)\Big\} \end{aligned}$$

for some $a_1, b_1, \ldots, a_k, b_k \in X$, where $\lambda_i : X \to P$ ($i = 1, \ldots, k$) are arbitrary functions and $X$ is a nonempty set.

On the other hand, the next result follows from the preceding statements.

**Corollary 2.** *Let $L$ be a lattice with the order relation $\preccurlyeq$. Then for some arbitrary mappings $p : X \to L$ and $q : X \to L$ ($X$ is an arbitrary nonempty set) the following equality holds*

$$\max_{x,y \in X} \inf \{p(x), q(y)\} = \min_{x,y \in X} \sup \{p(x), q(y)\}$$

*if and only if the following inequalities hold*

$$\inf \{p(x), q(y)\} \preccurlyeq \inf \{p(x_0), q(y_0)\} = \sup \{p(r_0), q(z_0)\} \preccurlyeq \sup \{p(x), q(y)\}$$

*for some $x_0, y_0, r_0, z_0 \in X$ and for all $x, y \in X$.*

We note, in the preceding statements (as in Corollary 23) we can have defined the preceding functions $p, q : X \to L$ and on different sets, in the sense that $p : X \to L$ and $q : Y \to L$ ($X$ and $Y$ are arbitrary nonempty sets). Then the preceding statements hold too. In this sense, for some arbitrary functions $f_i : X_i \to L$ ($i = 1, \dots, k$) the following equality holds

$$\max_{x_1 \in X_1, \dots, x_k \in X_k} \inf \Big\{ f_1(x_1), \dots, f_k(x_k) \Big\} = \min_{x_1 \in X_1, \dots, x_k \in X_k} \sup \Big\{ f_1(x_1), \dots, f_k(x_k) \Big\}$$

if and only if the following inequalities hold

$$\inf \Big\{ f_1(x_1), \dots, f_k(x_k) \Big\} \preccurlyeq \inf \Big\{ f_1(a_1), \dots, f_k(a_k) \Big\} =$$
$$= \sup \Big\{ f_1(b_1), \dots, f_k(b_k) \Big\} \preccurlyeq \sup \Big\{ f_1(x_1), \dots, f_k(x_k) \Big\}$$

for some $a_i, b_i \in X_i$ ($i = 1, \dots, k$) and for all $x_i \in X_i$ ($i = 1, \dots, k$).

In this part of this section, we show that the existence of a separation in the preceding sense, is essential for applications of the preceding statements. This is a separation for the preceding equalities of Sup-Inf (Min-Max) type.

In this sense we give a **characterization of general variational equality**. It results in the following.

**Theorem 4** (Statement of Separation))**.** *Let $L$ be a lattice with the order relation $\preccurlyeq$, and with local comparability. Then for some arbitrary mappings $p : X \to L$ and $q : Y \to L$ ($X$ and $Y$ are two arbitrary nonempty sets) the following equality holds*

$$\text{(IS)} \qquad \mathrm{Max}_{x \in X}\, p(x) = \mathrm{Min}_{y \in Y}\, q(y)$$

*if and only if there exists a decreasing function $g : L^2 \to L$ such that the following inequalities hold*

$$\text{(PQ)} \qquad p(x) \preccurlyeq g(p(x), q(y)) \preccurlyeq q(y) \quad \textit{for all} \quad x \in X \ \textit{and}\ y \in Y,$$

*and if there is $\xi \in L$ such that the $\xi \cap p(X)$ and $\xi \cap q(Y)$ are nonempty sets.*

*Proof. Necessity.* Let the inequalities (PQ) hold and from the conditions let, exist points $x_0 \in X$ and $y_0 \in Y$ such that $\xi = p(x_0) = q(y_0)$. Thus, we obtain the following inequalities and equality of the form

$$\inf\{p(x), q(y), g(p(x), q(y))\} \preccurlyeq \xi = g(\xi, \xi) \preccurlyeq \sup\{p(x), q(y), g(p(x), q(y))\}$$

for some $x_0 \in X$ and $y_0 \in Y$, and for all $x \in X$ and $y \in Y$. This means, from Theorem 42 and from (PQ), that the equality (MM) holds, which gives the equality (IS) of this statement.

*Sufficiency.* Assume that equality (IS) holds. Thus, there is $\xi \in L$ such that $p(x) \preccurlyeq \xi \preccurlyeq q(y)$ for all $x \in X$ and $y \in Y$, where $p(x_0) = q(y_0) = \xi$ for some $x_0 \in X$ and $y_0 \in Y$. If the decreasing function $g : L^2 \to L$ defined by $g(s, t) = \xi$, then, directly, we obtain inequalities (PQ). The proof is complete. □

At the end of this section, based on the preceding statements, we have the following fact as an immediate consequence.

**Corollary 3.** *Let $P$ be a set totally ordered by the order relation $\preccurlyeq$, and let $g : P^2 \to P$ be a decreasing mapping. Then the following equality holds*

$$\max_{\xi \preccurlyeq x} \min_{y \preccurlyeq \xi} g(x, y) = \min_{y \preccurlyeq \xi} \max_{\xi \preccurlyeq x} g(x, y)$$

*if and only if there is* $\xi \in P$ *such that* $g(\xi,\xi) = \xi$.

**Characterizations of Sup-Inf Equalities.** In this part we continue the considerations of sup-inf equalities. We prove some further characterizations of the preceding equalities on conditionally complete partially ordered sets. In this sense, $P$ is *conditionally complete* if every nonempty subset of $P$ with upper bounds has its supremum. We begin with the following statement.[5]

**Theorem 5.** *Let* $S$ *be a conditionally complete lattice by the order relation* $\preccurlyeq$, $f : X \to S$ *(*$X$ *is a nonempty set) has a minimum and* $g : Y \to S$ *(*$Y$ *is a nonempty set) has a maximum. If* $G : f(X) \times g(Y) \to S$, *then the equality*

$$\min_{x\in X}\max_{y\in Y}\sup\Big\{f(x), g(y), G(f(x), g(y))\Big\} = \max_{y\in Y}\min_{x\in X}\inf\Big\{f(x), g(y), G(f(x), g(y))\Big\} \tag{32}$$

*holds, if and only if for any two finite sets* $\{x_1, x_2, \ldots, x_n\} \subset X$ *and* $\{y_1, y_2, \ldots, y_m\} \subset Y$ *there exist* $x_0 \in X$ *and* $y_0 \in Y$ *such that*

$$\sup\Big\{f(x_0), g(y_k), G(f(x_0), g(y_k))\Big\} \preccurlyeq \inf\Big\{f(x_i), g(y_0), G(f(x_i), g(y_0))\Big\} \tag{33}$$

*for all* $i = 1, \ldots, n$ *and for all* $k = 1, \ldots, m$.

As an immediate consequence of the preceding Theorem 46 we obtain the following statement on topological spaces.

**Theorem 6.** *Let* $X$ *and* $Y$ *be two compact Hausdorff spaces,* $f : X \to \mathbb{R}$, $g : Y \to \mathbb{R}$ *and* $G : f(X) \times g(Y) \to \mathbb{R}$. *Suppose* $A(x,y) := \min\{f(x), g(y), G(f(x), g(y))\}$ *is lower semi-continuous on* $X$ *for every* $y \in Y$ *and* $B(x,y) := \max\{f(x), g(y), G(f(x), g(y))\}$ *is upper semi-continuous on* $Y$ *for every* $x \in X$. *Then the equality*

$$\min_{x\in X}\max_{y\in Y}\max\Big\{f(x), g(y), G(f(x), g(y))\Big\} = \max_{y\in Y}\min_{x\in X}\min\Big\{f(x), g(y), G(f(x), g(y))\Big\} \tag{34}$$

[5] **Polemics between M. Fréchet and J. von Neumann.** It is well known that Émile Borel in 1921, 1924, and 1927 has been published three papers on French language which are on English language (in translation and editorial work by M. Fréchet published in journal: Econometrica, **21** (1953), 97-117; under the following titles: *The theory of play and integral equations with skew symmetric kernels*, 97-100; *On games that involve chance and the skill of the players*, 101-115; and *On systems of linear forms of skew symmetric determinant and the general theory of play*, 116-117.

In connection with this, M. Fréchet is continually to establish that É. Borel is to conceive of games theory. In this context he has been publish two papers: *Émile Borel, Initiator or the theory of psychological games and its application*, Econometrica, **21** (1953), 9-96; and *Ecommentary of the three notes on Émile Borel*, Econometrica, **21** (1953), 118-124.

Take the preceding papers is to react J. von Neumann which is implacability to establish own to be right on advantage in authorship for concept of games theory. In this sense see paper: J. von Neumann, *Communication on the Borel notes*, Econometrica, **21** (1953), 124-125.

For von Neumann's contribution to games theory see: H.W. Kuhn and A.W. Tucker, *John von Neumann's work in the theory of games and mathematical econometrics*, Bull. Amer. Math. Soc., **64** (1958), 100-122.

*holds, if and only if for any two finite sets* $\{x_1, x_2, \dots, x_n\} \subset X$ *and* $\{y_1, y_2, \dots, y_m\} \subset Y$ *there exists* $x_0 \in X$ *and* $y_0 \in Y$ *such that*

$$\text{(33')}\quad \max\Big\{f(x_0), g(y_k), G(f(x_0), g(y_k))\Big\} \leqslant \min\Big\{f(x_i), g(y_0), G(f(x_i), g(y_0))\Big\}$$

*for all* $i = 1, 2, \dots, n$ *and for all* $k = 1, 2, \dots, m$.

A brief proof of this statement based on the preceding facts may be found in 1993 by Tasković. Also see: Tasković [2001].

**Sup-Inf Inequalities.** We give now some immediate applications of the preceding statements to sup-inf inequalities.

As an immediate consequence of Lemma 4a we obtain the following inequalities.

**Lemma 4.** *Let* $P$ *be a total ordered set by the order relation* $\preccurlyeq$, *and let* $g : P^2 \to P$ *be a decreasing mapping. If for some arbitrary mapping* $f : P^2 \to P$ *is* $f(\xi, \xi) \preccurlyeq \xi$ *and* $f(\xi, \xi) \preccurlyeq g(\xi, \xi)$, *then*

$$\text{(Sf)}\qquad f(\xi,\xi) \preccurlyeq \max\Big\{p(x), q(y), g(p(x), q(y))\Big\} \quad \text{for all} \quad x, y \in X,$$

*where* $p, q : X \to P$ *and* $X$ *is an arbitrary nonempty set.*

Quantifying the preceding assertion (Sf) we obtain the following conclusion that $f(\xi, \xi) \preccurlyeq \xi$ and $f(\xi, \xi) \preccurlyeq g(\xi, \xi)$ implies

$$f(\xi,\xi) \preccurlyeq \min_{x,y\in X} \max\Big\{p(x), q(y), g(p(x), q(y))\Big\}.$$

*Proof.* Let $\lambda = \max\{p(x), q(y)\}$ where the elements $x \in X$ and $y \in Y$ are arbitrarily chosen. If $f(\xi, \xi) \preccurlyeq \lambda$, then (Sf) obviously holds. If $\lambda \preccurlyeq f(\xi, \xi)$, then $f(\xi, \xi) \preccurlyeq g(\xi, \xi) \preccurlyeq g(p(x), q(y))$ and (Sf) holds too. $\square$

In connection with this, we now obtain the fundamental fact of this section, which is essential for inequalities.

**Theorem 7** (Sup-Inf Inequality)**.** *Let* $L := (L, \preccurlyeq)$ *be a lattice with zero and unit, and let* $A, B : X \times Y \to L$ *(*$X$ *and* $Y$ *are arbitrary nonempty sets). Then for arbitrary mappings* $a, c : X \to L$ *and* $b, d : Y \to L$ *with* $a(x), b(y), A(x, y) \preccurlyeq c(x), d(y), B(x, y)$ *for all* $x \in X$ *and* $y \in Y$, *the following inequality holds*

$$\text{(IN)}\qquad \inf_{x\in X, y\in Y} \sup\big\{a(x), b(y), A(x,y)\big\} \preccurlyeq \sup_{x\in X, y\in Y} \inf\big\{c(x), d(y), B(x,y)\big\}$$

*if and only if the following inequality holds*

$$\text{(OI)}\quad \sup\{a(x), b(y), A(x,y)\} \preccurlyeq \inf\{c(x), d(y), B(x,y)\} \quad \text{for all} \quad x \in X, y \in Y.$$

As an immediate consequence of the preceding statement we obtain the following statement.

**Theorem 8.** *Let* $(L, \preccurlyeq)$ *be a lattice with zero and unit, and let* $A, B : X \times Y \to L$ *(*$X$ *and* $Y$ *are arbitrary nonempty sets). Then for arbitrary mappings* $a, c : X \to L$ *and* $b, d : Y \to L$ *with* $a(x), b(y), A(x, y) \preccurlyeq c(x), d(y), B(x, y)$ *for all* $x \in X$ *and* $y \in Y$, *the following inequality holds*

$$\inf_{x\in X, y\in Y} \sup\big\{a(x), b(y), A(x,y)\big\} \preccurlyeq \sup_{x\in X, y\in Y} \sup\big\{c(x), d(y), B(x,y)\big\}$$

*if and only if the following inequality holds*

(S) $\sup\{a(x), b(y), A(x,y)\} \preccurlyeq \sup\{c(x), d(y), B(x,y)\}$ *for all* $x \in X, y \in Y$.

At the end of this section, we give a separation of statement for separation of the preceding inequalities.

**Theorem 9** (Separation of Inequalities)**.** *Let $L$ be a conditionally complete lattice with the order relation $\preccurlyeq$, and let the functions $c : X \to L$ and $b : Y \to L$ ($X$ and $Y$ are two arbitrary nonempty sets) satisfy the inequality $b(y) \preccurlyeq c(x)$ for all $x \in X$ and $y \in Y$. Then the following inequality holds*

$$\text{(NT)} \qquad \mathrm{Inf}_{y\in Y}\, b(y) \preccurlyeq \mathrm{Sup}_{x\in X}\, c(x)$$

*if and only if there exist functions $A, B : X \times Y \to L$, $a : X \to L$ and $d : Y \to L$ such that the following inequalities hold*

$$\text{(NI)} \qquad a(x) \preccurlyeq A(x,y) \preccurlyeq b(y) \preccurlyeq c(x) \preccurlyeq B(x,y) \preccurlyeq d(y)$$

*for all $x \in X$ and for all $y \in Y$.*

As a direct consequence of Theorem 49 we obtained in 1972 the so-called well known *Inequality of Fan in the form as in the problem 4.14*. Also, directly, from Theorems 48 and 49 we have statements of this form by: Granas-Liu (Andrzej Granas, Fon-Che Liu) [1986], Yen [1981], Kindler (Jurgen Kindler) [1983], Gale (D. Gale) [1955], Nikaido (H. Nikaido) [1956], Debreu (G. Debreu) [1959], and Ptak (Vlastimil Ptak) [1959]. For further facts see: Tasković [2005].

**Characterizations of Sup-Inf Inequalities.** In connection with the preceding facts, in this section we give further characterizations of the preceding sup-inf inequalities.

**Theorem 10.** *Let $S$ be a conditionally complete lattice by the order relation $\preccurlyeq$, and $A, B : X \times Y \to S$ ($X$ and $Y$ are arbitrary nonempty sets). Suppose that $A$ and $a : X \to S$ have minimums on $X$, and that $B$ and $d : Y \to S$ have maximums on $Y$. If $\rho : X \to S$ and $b : Y \to S$, then the inequality*

$$(35) \qquad \min_{x\in X} \sup_{y\in Y} \sup\{a(x), b(y), A(x,y)\} \preccurlyeq \max_{y\in Y} \inf_{x\in X} \inf\{\rho(x), d(y), B(x,y)\}$$

*holds, if and only if for any two finite sets $\{x_1, x_2, \dots, x_n\} \subset X$ and $\{y_1, y_2, \dots, y_m\} \subset Y$ there exist $x_0 \in X$ and $y_0 \in Y$ such that*

$$(36) \qquad \sup\{a(x_0), b(y_k), A(x_0, y_k)\} \preccurlyeq \inf\{\rho(x_i), d(y_0), B(x_i, y_0)\}$$

*for all $i = 1, 2, \dots, n$ and for all $k = 1, 2, \dots, m$.*

As an immediate consequence of Theorem 10, we obtain the following statement on topological spaces.

**Theorem 11.** *Let $X$ and $Y$ be two compact Hausdorff spaces, let $A, B : X \times Y \to \mathbb{R}$, let $\rho, a : X \to \mathbb{R}$ and $b, d : Y \to \mathbb{R}$. Suppose that $(x,y) \mapsto \min\{a(x), b(y), A(x,y)\}$ is lower semi-continuous on $X$ for every $y \in Y$ and $(x,y) \mapsto \max\{\rho(x), d(y), B(x,y)\}$ is upper semi-continuous on $Y$ for every $x \in X$. Then the inequality*

$$\min_{x\in X} \max_{y\in Y} \max\{a(x), b(y), A(x,y)\} \leqslant \max_{y\in Y} \min_{x\in X} \min\{\rho(x), d(y), B(x,y)\}$$

*holds, if and only if for any two finite sets* $\{x_1, x_2, \ldots, x_n\} \subset X$ *and* $\{y_1, y_2, \ldots, y_m\} \subset Y$ *there exist* $x_0 \in X$ *and* $y_0 \in Y$ *such that*

$$\max\left\{a(x_0), b(y_k), A(x_0, y_k)\right\} \leqslant \min\left\{\rho(x_i), d(y_0), B(x_i, y_0)\right\}$$

*for all* $i = 1, 2, \ldots, n$ *and for all* $k = 1, 2, \ldots, m$.

**Characterizations of Ky Fan type.** In this part we continue the considerations of some sup-inf inequalities of Ky Fan type. We begin with the following essential statement.

**Theorem 12.** *Let* $S$ *be a conditionally complete lattice by the order relation* $\preccurlyeq$, *and* $f, g : X \times Y \to S$ *(*$X$ *and* $Y$ *are nonempty sets) such that* $x \mapsto f(x, y)$ *has a minimum on* $X$ *and* $y \mapsto g(x, y)$ *has a maximum on* $Y$. *Then the inequality*

$$\min_{x \in X} \sup_{y \in Y} f(x, y) \preccurlyeq \max_{y \in Y} \inf_{x \in X} g(x, y) \tag{37}$$

*holds, if and only if for any two finite sets* $\{x_1, x_2, \ldots, x_n\} \subset X$ *and* $\{y_1, y_2, \ldots, y_m\} \subset Y$ *there exist* $x_0 \in X$ *and* $y_0 \in Y$ *such that*

$$f(x_0, y_k) \preccurlyeq g(x_i, y_0) \quad \text{for} \quad 1 \leq i \leqslant n,\ 1 \leq k \leq m. \tag{38}$$

As an immediate consequence of the preceding Theorem 12 we obtain the following statement on topological spaces for $S = \mathbb{R}$.

**Theorem 13.** *Let* $X$ *and* $Y$ *be two compact Hausdorff spaces, and* $f, g : X \times Y \to \mathbb{R}$ *such that* $x \mapsto f(x, y)$ *is lower semi-continuous on* $X$ *for every* $y \in Y$ *and* $y \mapsto g(x, y)$ *is upper semi-continuous on* $Y$ *for every* $x \in X$. *Then the inequality*

$$\min_{x \in X} \sup_{y \in Y} f(x, y) \leqslant \max_{y \in Y} \inf_{x \in X} g(x, y) \tag{39}$$

*holds, if and only if for any two finite sets* $\{x_1, x_2, \ldots, x_n\} \subset X$ *and* $\{y_1, y_2, \ldots, y_m\} \subset Y$ *there exist* $x_0 \in X$ *and* $y_0 \in Y$ *such that*

$$f(x_0, y_k) \leq g(x_i, y_0) \quad \text{for} \quad 1 \leq i \leqslant n,\ 1 \leq k \leq m. \tag{40}$$

In connection with the preceding facts, as an immediate consequence of Theorem 13, we obtain the following fundamental statement of Ky Fan for $f = g$.

**Theorem 14** (Ky Fan, [1953]). *Let* $X$ *and* $Y$ *be two compact Hausdorff spaces, and let* $f : X \times Y \to \mathbb{R}$ *such that* $x \mapsto f(x, y)$ *is lower semi-continuous on* $X$ *for every* $y \in Y$ *and* $y \mapsto f(x, y)$ *is upper semi-continuous on* $Y$ *for every* $x \in X$. *Then the equality*

$$\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y) \tag{41}$$

*holds, if and only if for two finite sets* $\{x_1, x_2, \ldots, x_n\} \subset X$ *and* $\{y_1, y_2, \ldots, y_m\} \subset Y$ *there exist* $x_0 \in X$ *and* $y_0 \in Y$ *such that*

$$f(x_0, y_k) \leq f(x_i, y_0) \quad \text{for} \quad 1 \leq i \leqslant n,\ 1 \leq k \leq m. \tag{42}$$

We notice that various generalizations of Von Neumann's minimax theorem have been given by several authors. In all these theorems, the structure of linear space is always present. This result of Ky Fan is for first time a form of minimax theorem which involves no linear space.

**Transversal points.** In connection with the preceding, in this part we continue the study of the preceding minimax problems. Further on we consider concept of transversal points[6] for the mapping $f$ of a nonempty set $X$ into partially ordered set $P$. A map $f$ of a nonempty set $X$ into partially ordered set $P$ has a *transversal point* $\zeta \in P$ if there is a decreasing function $g : P^2 \to P$ such that the following equality holds

$$\text{(T)} \qquad \begin{aligned} &\max_{x,y\in X} \min \Big\{ f(x), f(y), g(f(x), f(y)) \Big\} = \\ &\min_{x,y\in X} \max \Big\{ f(x), f(y), g(f(x), f(y)) \Big\} := \zeta. \end{aligned}$$

On the other hand, in 1988 we investigated the concept od fixed apices for a mapping $f$ of a set $X$ into itself. A map $f$ of a set $X$ to itself has a *fixed apex* $u \in X$ if for $u \in X$ there is $v \in X$ such that $f(u) = v$ and $f(v) = u$. The points $u, v \in X$ are called *fixed apices* of $f$ if $f(u) = v$ and $f(v) = u$. In this sense, a nonempty set $X$ is *apices set* if each of its points is an apex of some mapping $T : X \to X$. If $T : \mathbb{S}^n \to \mathbb{S}^n$ is the map such that $Tx = -x$ for $x \in \mathbb{S}^n$, then $\mathbb{S}^n$ is an apices set.

Otherwise, a function $f : X \to P$ has a *SI-transversal point* $\zeta$ if the preceding equality (T) holds with sup and inf instead of max and min, respectively. If the preceding equality (T) holds for points $x, -x \in X$ ($X$ is a linear space) $\zeta$ is *A-transversal point*; more generally $\zeta$ is *R-transversal point* for $f : X \to P$ if the equality (T) holds for points $x, Tx \in X$. A function $f : X \to P$ ($X$ is a linear

[6]**The genus of topological space**. In the sense of further extensions of the preceding facts we have the following term of genus. Let $X$ be a topological space and $T : X \to X$ a fixed-point free involution on $X$, i.e., $T^2 = \mathrm{id}$. The **genus of space** $X$ denoted by $g(X;T)$ is the smallest positive integer $m$ for which there can be found $m$ closed sets $A_1, A_2, \ldots, A_m$ such that (a) $A_i \cap TA_i = \varnothing$ and (b) $X = \cup_{i=1}^{m}(A_i \cup TA_i)$. This invariant is closely related to the notion of the Lusternik-Schnirelman category and to various extensions of the Borsuk antipodal theorem. This invariant had been explicitly considered by C.T. Yang [1954] and M.A. Krasnoselskij [1955a].

The following statements are equivalent from C.T. Yang [1955]: (I) $g(X;T) \geqslant n+1$; (II) *there is no map* $f : X \to \mathbb{S}^{n-1}$ *such that* $f(Tx) = -f(x)$ *for all* $x \in X$; (III) *if* $X$ *is covered by* $(n+1)$*-closed sets at least one of them contains a pair* $x, Tx$; (IV) *if* $f : X \to \mathbb{R}^n$*, there is a point* $x \in X$ *such that* $f(x) = f(Tx)$. Note that in case $X = \mathbb{S}^n$ and $T$ =antipodal map we obtain Lusternik-Schnirelman theorem, as and, adequate, Borsuk-Ulam theorem.

On the other hand, for an arbitrary involution $T$ on $\mathbb{S}^n$ we have $g(\mathbb{S}^n;T) \geqslant n+1$ from A.I. Fet [1954]. More generally, if $X$ is a compact with Čech homology $H_k(X;Z_2) \simeq H_k(\mathbb{S}^n;Z_2)$ for $k \leq n$, then for an arbitrary $T$, $g(X;T) \geqslant n+1$ from J.W. Jaworowski [1955].

More general or related results will be found in: C.T. Yang [1955], J.W. Jaworowski [1956] and M. Davies [1956]. See also: H. Hopf [1944], M.W. Hirsch [1944], M.A. Geraghty [1961] and P. Bacon [1966]. For uses of the genus in critical point theory see the book: *Topological methods in the theory of nonlinear integral equations* by M.A. Krasnoselskij [1956].

The notion of a genus can be consider in the following more general sense: Let $X$ be a topological space on which a finite group $G$ acts without fixed points. The **genus** $g(X;G)$ of $X$ with respect to $G$ is the smallest positive integer $n$ for which there can be found $n$ closed sets $A_1, A_2, \ldots, A_n$ such that: (a) $A_i \cap g(A_i) = \varnothing$ unless $g = 1$, and (b) $X = \cup\{g(X_i) : g \in G;\ i = 1, \ldots, n\}$. This invariant has been considered for $G = Z_p$ by M.A. Krasnoselskij [1955] and in full generality by A.S. Švarc [1957]. For further facts see: A.S. Švarc [1961] and H. Steinlein [1980].

space) has a pair of *antipodal points* $p, -p \in X$ if the following equality holds $f(p) = f(-p)$.

We note that from the second section, i.e., from Corollary 22, we obtain that the function $f(x) = \mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ has a transversal point $\zeta \in \mathbb{R}$ if and only if for some decreasing function $g : \mathbb{R}^2 \to \mathbb{R}$ we have $g(\zeta, \zeta) = \zeta$.

In connection with this, Borsuk-Ulam theorem is well-known in the following form: *Every continuous map $f : \mathbb{S}^n \to \mathbb{R}^n$ maps some pair of antipodal points into a single point.*

Results equivalent to the Lusternik, Schnirlman and Borsuk statement use the notions of extendability and homotopy in their formulation. For the convenience of the reader, and to establish the terminology, we recall the relevant definitions. By space, we understand a Hausdorff space; unless otherwise specifically stated, a map is continuous transformation.

We now prove Borsuk's antipodal theorem and also show that it is equivalent to various geometric results about the $n$-sphere. Further on, we continue to study the interaction of covering, antipodal and transversal points.

**Theorem 15.** *Let $\mathbb{S}^n$ denote the $n$-sphere. Then the following statements are equivalent in the following sense as:*

(a) (Lusternik-Schnirelman-Borsuk theorem). *In any closed covering $\{M_1, \ldots, M_{n+1}\}$ of $\mathbb{S}^n$ by $(n+1)$-sets, at least one set $M_i$ must contain a pair of antipodal points.*

(b) (Borsuk antipodal theorem). *An antipodal-preserving map $f : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is not nullhomotopic.*

(c) (Borsuk-Ulam type theorem). *Every continuous map $f : \mathbb{S}^n \to \mathbb{R}^n$ sends at least one pair of antipodal points to the same point.*

(d) (Transversal point theorem). *Every continuous map $f : \mathbb{S}^n \to \mathbb{R}$ has at least one $A$-transversal point.*

(e) (Antipodal point theorem). *Every continuous map $f : \mathbb{S}^n \to \mathbb{R}^n$ has at least one pair of antipodal points.*

In connection with the former results of Lusternik, Schnirelman, Borsuk and Theorem 53, as an immediate consequence we obtain the following fact.

**Corollary 4.** *Let $\mathbb{S}^n$ denote the $n$-sphere. Then the following statements are equivalent in the following form as:*

(a) (Borsuk-Ulam Theorem). *Every continuous map $f : \mathbb{S}^n \to \mathbb{R}^n$ sends at least one pair of antipodal points to the same point.*

(b) (Transversal point theorem). *Every continuous map $f : \mathbb{S}^n \to \mathbb{R}$ has at least one $A$-transversal point.*

(c) (Antipodal point theorem). *Every continuous map $f : \mathbb{S}^n \to \mathbb{R}$ has at least one pair of antipodal points.*

On the other hand, analogous to the preceding statements, we obtain the following extension of the former results on some apices sets.

**Theorem 16.** *Let $X$ be a topological space and $T : X \to X$ a fixed point free involution on $X$, i.e., $T^2 = \mathrm{id}$. Then the following statements are equivalent:*

(a) *In any closed covering $\{M_1, \ldots, M_{n+1}\}$ of $X$ by $(n+1)$-sets, at least one of them must contain a pair of points $x, Tx \in X$.*

(b) *Every continuous map $f : X \to \mathbb{R}$ has at least one pair of points $p, Tp \in X$ such that $f(p) = f(Tp)$.*
(c) *Every continuous map $f : X \to \mathbb{R}$ has at least one $R$-transversal point.*

Note that for Theorem 53a in case $X = \mathbb{S}^n$ and $T =$ antipodal map, (a) is the Lusternik-Schnirelman-Borsuk theorem, (b) is the Antipodal point theorem and (c) is the Transversal point theorem.

In connection with the former facts on transversal points, we have the following extensions. A map $f : X \to P$ ($X$ is an arbitrary nonempty set and $P$ is a poset) has a *general transversal point* $\xi \in P$ if there is a decreasing function $g : P^k \to P$ ($k$ is a fixed positive integer) such that the following equality holds

$$\text{(T')} \quad \begin{aligned} &\max_{x_1,\dots,x_k \in P} \min\Big\{f(x_1), \dots, f(x_k), g(f(x_1), \dots, f(x_k))\Big\} = \\ &= \min_{x_1,\dots,x_k \in P} \max\Big\{f(x_1), \dots, f(x_k), g(f(x_1), \dots, f(x_k))\Big\} := \xi. \end{aligned}$$

From the former part, i.e., from Theorem 2, the case (Uk), we obtain that the function $f : X \to P$ has general transversal point if and only if

$$f(a_1) = \ldots = f(a_k) := \xi = g(\xi, \dots, \xi) \quad \text{for some} \quad a_1, \dots, a_k \in P.$$

**Roots of Algebraic Equations.** We note that, by the application of Lemma 1 (in fact of (U), i.e., of Lemma 2), one can simultaneously obtain the upper and lower bounds of the roots of the equation

$$g\left(\frac{1}{x}, \dots, \frac{1}{x}\right) = x \quad \text{for} \quad x \in \mathbb{R}_+, \tag{43}$$

where $g : (\mathbb{R}_+)^n \to \mathbb{R}_+$ is a nondecreasing function.

As an immediate consequence of Theorem 2, the case (Uk), we obtain the following statement, which is essential in algebra!

**Theorem 17** (Tasković, [1978]). *A point $\zeta \in \mathbb{R}_+ := (0, +\infty)$ is the root of equation of the form* (43) *if and only if the following equality holds*

$$\begin{aligned} \zeta = &\max_{\lambda_1,\dots,\lambda_n \in \mathbb{R}_+} \min\left\{\lambda_1, \dots, \lambda_n,\ g\left(\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}\right)\right\} = \\ = &\min_{\lambda_1,\dots,\lambda_n \in \mathbb{R}_+} \max\left\{\lambda_1, \dots, \lambda_n,\ g\left(\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}\right)\right\}. \end{aligned}$$

In connection with the preceding facts about transversal points, from Theorem 2, we obtain that the equation (43) has a root $\zeta \in \mathbb{R}_+$ if and only if the point $\zeta$ is a general transversal point of the function $f(x) = \mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$.

**Theorem 18** (Tasković, [1984]). *Let $I_1, \dots, I_n$ be indices sets and $\theta_{i_j} \geqslant 0$ be real numbers which satisfy the following condition*

$$\sum_{i_j \in I_j} \theta_{i_j} = j - t \quad \textit{for} \quad j = 1, \dots, n \quad \textit{and} \quad 0 < t < 1.$$

*Then $\zeta \in \mathbb{R}_+$ is the root of the following algebraic equation which is given in the form $x^t = a_1 x^{t-1} + \ldots + a_n x^{t-n}$ $((a_1, \dots, a_n) \neq (0, \dots, 0))$ if and only if the* *following equality holds*

$$\max_{M_{i_j}}\min\left\{M_{i_j},\left(\sum_{j=1}^{n}\frac{a_j}{\prod\limits_{i_j\in I_j}M_{i_j}^{\theta_{i_j}}}\right)^{1/t}\right\}=\min_{M_{i_j}}\max\left\{M_{i_j},\left(\sum_{j=1}^{n}\frac{a_j}{\prod\limits_{i_j\in I_j}M_{i_j}^{\theta_{i_j}}}\right)^{1/t}\right\}:=\zeta.$$

*Proof.* In order to prove this statement we may choose the function $g:\mathbb{R}_+^n\to\mathbb{R}_+$ ($n$ is a fixed positive integer) defined by

$$g(x_1,\dots,x_n):=\left(\sum_{j=1}^{n}\frac{a_j}{\prod\limits_{i_j\in I_j}M_{i_j}^{\theta_{i_j}}}\right)^{1/t}\qquad\text{for}\quad x_1:=M_{i_1},\dots,x_n:=M_{i_n}\in\mathbb{R}_+,$$

and then apply Theorem 2, the case (Uk). The proof is complete. □

**Furcate points.** In connection with the transversal points, continue the study of the former minimax problems, in the part we consider and some other concepts of points for the mapping $f$ of a nonempty set $X$ into a partially ordered set $P$. A map $f:X\to P$ has a **furcate point** $\xi\in P$ if for some function $T:X\to X$ the following equation holds

$$\text{(FP)}\qquad \max_{x,y\in X}\min\big\{f(x),f(Ty)\big\}=\min_{x,y\in X}\max\big\{f(x),f(Ty)\big\}:=\xi.$$

Otherwise, a function $f:X\to P$ has a **SI-furcate point** if the preceding equality (FP) holds when instead max and min standing sup and inf, respectively. If the preceding equality (FP) holds for points $x,-x\in X$ ($X$ is a linear space) then the point like this is said to be a **A-furcate point**; or general we call of **R-furcate point** for $f:X\to P$ if the equality (FP) holds for pair points $x,Tx\in X$.

From the second section, i.e., from Tasković [1990], we obtain that the function $f:X\to L$ ($X$ is an arbitrary nonempty set and $L:=(L,\preccurlyeq)$ is a lattice) has a SI-furcate point if and only if the following inequalities hold

$$\inf\big\{f(x),f(Ty)\big\}\preccurlyeq f(x_0)=f(Ty_0)\preccurlyeq\sup\big\{f(x),f(Ty)\big\},$$

for some $x_0,y_0\in X$ and for all $x,y\in X$. Thus, if $f:X\to L$ has a R-furcate point then $f$ has at least one pair of points, $p,Tp\in X$ such that $f(p)=f(Tp)$. Reversed does not hold. In this sense, on the Figure 10, for the mapping $f$ of complete lattice $I$ into itself is $f(p)=f(Tp)$ for some $p\in I$, but $f$ does not have furcate points.

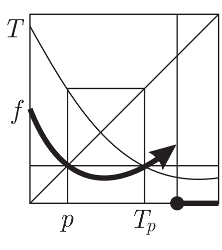


Figure 10

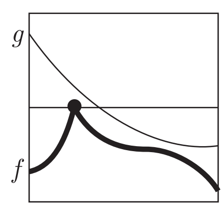


Figure 11

For two mappings $f : X \to P$ and $g : Y \to P$ ($X$ and $Y$ are arbitrary nonempty sets and $P$ is a partially ordered set) we have common (coincidence) furcate points.

Namely, two mappings $f : X \to P$ and $g : Y \to P$ have a **coincidence furcate point** $\xi \in P$, if the following equality holds

$$\max_{x\in X, y\in Y} \min \big\{f(x), g(y)\big\} = \min_{x\in X, y\in Y} \max \big\{f(x), g(y)\big\} := \xi.$$

Generally, the mappings $f_i : X_i \to P$ $(i = 1, \ldots, k)$ have a **coincidence furcate point** $\xi \in P$ if the following equality holds

$$\max_{x_1\in X_1,\ldots,x_k\in X_k} \min \big\{f(x_1), \ldots, f(x_k)\big\} = \min_{x_1\in X_1,\ldots,x_k\in X_k} \max \big\{f(x_1), \ldots, f(x_k)\big\} := \xi.$$

We notice, from T a s k o v i ć [1990], we obtain that the function $f : X \to P$ and $g : Y \to P$ have a coincidence furcate point if and only if the following inequalities hold

$$\min \big\{f(x), g(y)\big\} \preccurlyeq f(x_0) = g(y_0) \preccurlyeq \max \big\{f(x), g(y)\big\},$$

for some $x_0 \in X$, $y_0 \in Y$ and for all $x \in X$, $y \in Y$. In connection with this, we notice, there are some continuous functions $f, g : I \to I$ (Figure 11) which map compact interval into itself, but $f$ and $g$ have not any coincidence furcate points.

In our former consideration, we are to introduce concept of **general transversal point**, in the sense, that the function $f : X \to P$ ($X$ is a nonempty set and $P$ poset) has a **quasi transversal point** $\xi \in P$, if for some function $g : P^2 \to P$ the following equality holds

$$\max_{x,y\in X} \min \Big\{f(x), f(y), g(f(x), f(y))\Big\} = \min_{x,y\in X} \max \Big\{f(x), f(y), g(f(x), f(y))\Big\} := \xi.$$

Let $X$ and $P$ are two arbitrary nonempty sets. Two mappings $f : X \to P$ and $g : X \to P$ have a **coincidence general furcate point** $\xi \in P$ iff there exists a function $G : f(X) \times g(X) \to P$ such that

(R)
$$\min_{x\in X} \sup_{y\in X} \sup \Big\{f(x), g(y), G\big(f(x), g(y)\big)\Big\} = \max_{y\in X} \inf_{x\in X} \inf \Big\{f(x), g(y), G\big(f(x), g(y)\big)\Big\} := \xi.$$

We notice that this form of "object point" is very similar with the general transversal points, but indispensably different. The following result holds.

**Theorem 19** (Tasković, [1994]). *Let $P := (P, \preccurlyeq)$ be a conditionally complete set and let $G : P \times P \to P$, $f : X \to P$, and $g : X \to P$ be given mappings ($X$ is an arbitrary nonempty set). Suppose that $f(x)$ and $x \mapsto G\big(f(x), g(y)\big)$ have minimums and that $g(y)$ and $y \mapsto G\big(f(x), g(y)\big)$ have maximums, then the functions $f$ and $g$ have a coincidence general furcate point if and only if*

$$\text{(44)} \qquad \sup \Big\{f(x_0), g(y_k), G\big(f(x_0), g(y_k)\big)\Big\} \preccurlyeq \inf \Big\{f(x_i), g(y_0), G\big(f(x_i), g(y_0)\big)\Big\}$$

*for any two finite sets $\{x_1, \ldots, x_n\} \subset X$ and $\{y_1, \ldots, y_m\} \subset X$ for some $x_0, y_0 \in X$, and for all $i = 1, 2, \ldots, n$ and for all $k = 1, 2, \ldots, m$.*

On the other hand, we also have a second form for characterization of the coincidence general furcate points. In this sense, for the point of the form (R) the following equivalent booking holds:

(R')
$$\begin{aligned}&\min_{x\in X}\sup_{m\leqslant \operatorname{Card} X}\sup_{1\leq k\leqslant m}\sup\Big\{f(x),g(y_k),G\big(f(x),g(y_k)\big)\Big\}=\\&=\max_{y\in X}\inf_{n\leqslant \operatorname{Card} X}\inf_{1\leq i\leqslant n}\inf\Big\{f(x_i),g(y),G\big(f(x_i),g(y)\big)\Big\}:=\xi.\end{aligned}$$

A brief second proof of Theorem 19 may be found in T a s k o v i ć [2005]. Notice also, we that, in the special case for real functions, as an immediate consequence of Theorem 19, we directly obtain the following result.

**Theorem 20** (Tasković, [1994]). *Let $X$ be a compact Hausdorff space and let $G:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a given mapping. Suppose that $x\mapsto G\big(f(x),g(y)\big)$ and $f:X\to\mathbb{R}$ are two lower semicontinuous functions, and $y\mapsto G\big(f(x),g(y)\big)$ and $g:X\to\mathbb{R}$ are two upper semicontinuous functions. Then the following equality holds in the form*

(45)
$$\min_{x\in X}\max_{y\in X}\max\Big\{f(x),g(y),G\big(f(x),g(y)\big)\Big\}=\max_{y\in X}\min_{x\in X}\min\Big\{f(x),g(y),G\big(f(x),g(y)\big)\Big\}$$

*if and only if for two finite sets $\{x_1,\dots,x_n\}\subset X$ and $\{y_1,\dots,y_m\}\subset X$ there exist the points $x_0,y_0\in X$ such that*

(46) $$\max\Big\{f(x_0),g(y_k),G\big(f(x_0),g(y_k)\big)\Big\}\leqslant\min\Big\{f(x_i),g(y_0),G\big(f(x_i),g(y_0)\big)\Big\}$$

*for all indexes $1\leq i\leq n$ and for all indexes $1\leq k\leq m$.*

In the context of the preceding statement, two functions $f,g:X\to P$ ($X$ is an arbitrary nonempty set) have a **common general MM-forked point** $\xi\in P$ iff (45) holds. In this case, Theorem 20 is a characterization of this points for real functions on Hausdorff topological spaces.

Namely, two functions $f,g:X\to\mathbb{R}$ ($X$ is a compact Hausdorff topological space) have a common general MM-forked point $\xi\in\mathbb{R}$ if and only if the inequalities (46) hold.

**Open problem 1.** Given a new characterization of common general MM-forked points for two functions $f,g:X\to P$ defined on an arbitrary nonempty set $X$, where $P$ is a nonempty partially ordered set!?

As an immediate consequence of the preceding statements, we notice a result which is a characterization of common forked points in the form min-sup or max-inf.

**Theorem 21** (Tasković, [1994]). *Let $P:=(P,\preccurlyeq)$ be a conditionally complete set, and suppose that $f:X\to P$ has a minimum and $g:X\to P$ has a maximum. Then the following equality holds in the form*

(47) $$\min_{x\in X}\sup_{y\in X}\sup\Big\{f(x),g(y)\Big\}=\max_{y\in X}\inf_{x\in X}\inf\Big\{f(x),g(y)\Big\}$$

*if and only if for two finite sets $\{x_1,\dots,x_n\}\subset X$ and $\{y_1,\dots,y_m\}\subset X$ there exist $x_0,y_0\in X$ such that*

(47') $$\sup\Big\{f(x_0),g(y_k)\Big\}\preccurlyeq\inf\Big\{f(x_i),g(y_0)\Big\}$$

*for all indexes $i=1,\dots,n$ and for all indexes $k=1,\dots,m$.*

As in the preceding statements and in this case, adequately, we have some equivalent forms of the equality (47). For the further facts on this see T a s k o v i ć [2005].

**Further applications.** If $f$ is an odd function, then there is $x \in \mathbb{S}^1$ such that $f(x) = 0$. This is a generalization of the well known fact on odd functions. In this sense we have the following result.

**Proposition 1.** *Let $X$ be a nonempty apices set for the mapping $T : X \to X$ and let* $\operatorname{Card} X \geqslant$*continuum*:$= c$*. Then there is not a continuous mapping $f$ of $X$ into $\mathbb{S}^{n-1}$ such that $T(fp) = f(Tp)$ for every $p \in X$.*

As an immediate consequence of this statement we obtain the following characteristic fact in the preceding sense.

**Proposition 2.** *Let $X$ be a nonempty apices set for the mapping $T : X \to X$ and let* $\operatorname{Card} X \geqslant c$*. If $f$ is a continuous mapping of $X$ into $\mathbb{R}^n$ such that $f(Tp) = -f(p)$ for every $p \in X$, then there is $\xi \in X$ such that $f(\xi) = 0$.*

In connection with the preceding facts, further on we give the following illustrations. Suppose that $f$ is antipode-preserving. By applying a suitable rotation after $f$, we obtain a new antipode-preserving map of the same degree as $f$ and having a fixed point $A$. Then the antipode of $B$ is also fixed. We call this new map $f$ again as an Figure 12.

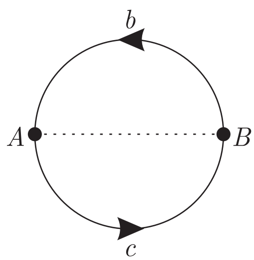


Figure 12

$$\begin{tikzcd} \mathbb{S}^n \arrow[r, "g"] \arrow[d, "p"'] \arrow[dr, "f\circ p = p\circ g" sloped] & \mathbb{S}^n \arrow[d, "p"] \\ \mathbb{P}^n \arrow[r, "f"'] & \mathbb{P}^n \end{tikzcd}$$

Figure 13

We proceed by induction on $n$. Split $\mathbb{S}^1$ into semicircles, $b$, $c$ using $B$ and its antipode $A$, i.e., $f$ has odd degree on $\mathbb{S}^1$.

Suppose that $\mathbb{P}^n$ is a real projective $n$-space, that $f : \mathbb{P}^n \to \mathbb{P}^n$ is continuous, and that $p : \mathbb{S}^n \to \mathbb{P}^n$ identifies antipodes. That is $f \circ p$ is homotopic to a map $p \circ g$, where $g : \mathbb{S}^n \to \mathbb{S}^n$ is continuous and, by the covering homotopy property, we may suppose $f \circ p = p \circ g$, as on Figure 13.

Also, if $f : \mathbb{S}^n \to \mathbb{R}^m$ $(m \leq n)$ is a continuous mapping such that the following equality holds in the form $f(-x) = -f(x)$ or only the following equality in the form

$$\frac{f(-x)}{|f(-x)|} = -\frac{f(x)}{|f(x)|},$$

i.e., $f(x)$ and $f(-x)$ have opposite directions when $\neq 0$, then $f$ vanishes somewhere. In this sense, if $f : \mathbb{S}^{2n} \to \mathbb{S}^{2n}$ is continuous, then either $f$ or $f^2$ has a fixed point; either $f$ has a fixed point or fixed apices, or interchanges a pair of points.

$$\begin{tikzcd}
\mathbb{S}^n \arrow[d, "p"'] \arrow[dr, "f"] & \\
\mathbb{P}^n \arrow[r, "\overline{f}"'] & \mathbb{S}^n
\end{tikzcd}$$

Figure 14

If $f$ collapses antipodal points, then $f$ splits through $\mathbb{P}^n$ as on Figure 14. Now, let $n$ be odd, $m$ any integer. Partition $\mathbb{S}^n$, as domain, into a descending sequence $\mathbb{S}^n \supset \mathbb{S}^{n-1} \supset \ldots \supset \mathbb{S}^1 \supset \mathbb{S}^0$ $(= A \cup B)$ of great spheres; partition $\mathbb{S}^n$ as range into $B$ and $\mathbb{S}^n \backslash B$. Reflect this map centrally to the "upper" hemisphere. The resulting map of $\mathbb{S}^n$ has a degree $2m$. For other facts of this type applications see: E. F. Whittlesey [1963], and Tasković [2005].

Also, *in place of sign, consider direction*! More precisely, if $F : \mathbb{S}^n \to \mathbb{R}^{n+1} \backslash \{0\}$ is continuous, and if $F$ never has the same direction at antipodal (transversal) points $x$ and $-x$ (Fig. 14), then $F$ is **essential**, i.e., the equation $F(x) = 0$ has a solution in the interior of the unit ball for each continuous extension of the boundary values.

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